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THE VALIDITY OF ASSUMPTIONS UNDERLYING CURRENT USES OF LANCHESTER ATTRITION RATES

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13. ABSTRACT *(Maximum 200 words)*
Part One of this study examines critically the various assumptions, results, and concepts that exist to date in the literature and scientific community concerning the relationships among the Lanchester, Stochastic Lanchester, and the General Renewal models of combat. Many of the prevailing understandings are shown to be errroneous. Conclusions and recommendations for further study are provided. In Part Two we collect, organize and reduce to common notation almost all known tabled results and curves of particular examples. These support the theoretical discussion.

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THE VALIDITY OF ASSUMPTIONS UNDERLYING CURRENT USES OF LANCHESTER ATTRITION RATES

STUDY GIST

THE REASON FOR PERFORMING THE STUDY was to show that many of the prevailing understandings concerning the relationships among the Lanchester, Stochastic Lanchester, and the General Renewal models of combat are erroneous and to collect, organize, and reduce to common notation almost all known tabled results and curves of particular examples.

THE PRINCIPAL RESULTS of this study were:

(1) All Lanchester model and Stochastic Lanchester model mean value equivalent pairs differ for all times except at possible crossing points. These differences may be very large.

(2) At least for the Square Law, the Lanchester model trajectories are neither a universal upper or lower bound of the Stochastic Lanchester model mean value trajectories.

(3) Even near time zero, the Lanchester and Stochastic Lanchester model mean value trajectories may differ considerably (they do not differ materially for the Square Law and the sequence of one-on-one duels version of the Linear Law).

(4) For the Linear Law, the Square Law, the Mixed Law and the Square Law with continuous reinforcements there is a Law of Large Numbers on suitably transformed spaces. However, for untransformed spaces this does not apply, for it can be shown that as the initial force sizes tend to infinity the differences between Lanchester model and Stochastic Lanchester model mean value trajectories tend to zero, or they may even tend to a constant or infinity.

(5) Blackwell's Theorum does not imply that individual combatants with general interfiring times tend to have negative exponentially distributed interfiring times. This is even more strongly the case for terminating processes.

(6) The Palm-Khintchine Theorem does not imply that superposing a large number of combatants with general interfiring times will yield a process with negative exponentially distributed interfiring times. This can only be approximately correct for large numbers and for very large interfiring time means. Again, the Theorem is only valid for non-terminating processes.

(7) Nonhomogeneous Poisson processes do not, in general, approximate general renewal processes.

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(8) The Stochastic Lanchester model process variances are generally quite significant and can be important for large force sizes, even near time zero. In addition, general renewal model process variances are significantly different that Stochastic Lanchester model process variances.

(9) The other Lanchester model measures, (a) expected number of survivors, (b) expected time duration of the battle, and (c) probability of winning are even less reliable predictors than the mean value trace.

(10) The basic assumptions of the Stochastic Lanchester models, as well as the general renewal model, cannot hold for large numbers of combatants.

THE MAIN ASSUMPTIONS were that:

(a) All pre-combat decisions have been made and the battle goes forward until terminated by the action itself or by tactical decisions.

(b) The true model which is to be approximated is termed a General Renewal model.

(c) Each marksman fires until he is killed or makes a kill.

(d) The ammunition supply is unlimited.

(e) All fire independently.

THE MAJOR RESTRICTION is that the general renewal model is a superposition of many terminating renewal processes.

THE SCOPE OF THE STUDY is to show that the basic assumptions of current combat models do not hold for large numbers of combatants.

THE STUDY OBJECTIVES are to show that many factors cause large scale battles to be a number of simultaneous and/or sequential smaller scale engagements and that use of current deterministic Lanchester models is incorrect.

THE BASIC APPROACH is two fold. First the Lanchester model applications are examined to show fallacies. Second all known tabled results and curves of particular examples are provided to support the theoretical discussion.

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NOTATION

Where multiple definitions are given, the context will make clear which one is used.

A - designation of one side in the combat, or
  - the event of a win by side A

$A_k(t)$ - rv, the number on the A side at time t, in the SL or GR model with ics (ka_0, kb_0) (when k = 1, subscript dropped), jointly distributed with $B_k(t)$

a - value of rv, $A_k(t)$

a_f - the number on side A at the time the A side loses (breaks and runs)

a_0 - the initial number on side A (at time zero)

B - designation of one side in the combat, or
  - the event of a win by side B

$B_k(t)$ - rv, the number on the B side at time t, in the SL or GR model with ics (ka_0, kb_0) (when k = 1, subscript dropped), jointly distributed with $A_k(t)$

b - value of rv, $B_k(t)$

b_f - the number on the B side at the time the B side loses (breaks and runs)

b_0 - the initial number on side B (at time zero)

cov(X,Y) - covariance of rvs X,Y

c - arbitrary constant

D - the event of a draw

d - arbitrary constant

E[X] - expected value of rv X

$\widehat{E[X]}$ - estimate of the mean of rv X

e - arbitrary constant

F_X^c - complementary distribution function of rv X

f - arbitrary constant

$f'(t) = \frac{df(t)}{dt}$

$f_X(x)$ - pdf of rv X

GR - the general renewal model

$g_i(t)$ - average reinforcement rate on the $i^{\underline{th}}$ side, i = A or B

ic - initial condition

ift - interfiring time

iid - independent, identically distributed (used with rvs)

k - positive integer, either an index, or
  - the number of the transition step in state-space

L - the Lanchester model

$m_A(t) = E[A(t)]$, marginal mean value function of $A(t)$

$m_B(t) = E[B(t)]$, marginal mean value function of $B(t)$

m_2 - portion of force engaged on A side in Springall model

$N(t)$ - rv, number of events which have occurred at time t in a GR renewal process

$n(t)$ - a value of the rv $N(t)$

$\overline{n(t)} = E[N(t)]$, mean value function of $N(t)$

n_2 - portion of force engaged on B side in Springall model

ned - negative exponentially distributed

$P(i)$ - probability that the i side wins, i = A or B

$P(i,t)$ - probability that the i side has won by time t, i = A or B

p - marksman's kill probability or
  $= \beta/(\alpha+\beta)$

$p(i,t)$ - pmf of rvs $A(t)$ and $B(t)$ respectively, i = a,b

$p(a,t|A)$ - conditional pmf of rv $A(t)$ given A wins by time t

$p(b,t|B)$ - conditional pmf of rv $B(t)$ given B wins by time t

p_i - the constant kill probability of all contestants on the $i^{\underline{th}}$ side, i=A or B

$p_i(t)$ - time-dependent kill probability of all contestants on the $i^{\underline{th}}$ side, i = A or B

$p_k(a,b,t) = P[A_k(t) = a,\ B_k(t) = b,\ ic(ka_0,kb_0)]$ (for k = 1 subscript is dropped)

pdf - probability density function

pmf - probability mass function

$q = 1 - p$, marksman's failure probability or
$= \alpha/(\alpha+\beta)$

rhs - right hand side of an equation

r_i - fixed individual firing rate for side i, i = A or B

$r_i(t)$ - time-dependent individual firing rate for side i, i = A or B

$r_i(a,b,t)$ - i's general kill rate (other side's attrition rate), i = A or B

$r(y) = f_Z(y)/F_Z^c(y)$, GR model individual instantaneous kill rate

rv - random variable

$S_A(t) = \alpha \sum_{a=1}^{a_0} ap(a,0,t)$

$S_B(t) = \beta \sum_{b=1}^{b_0} bp(0,b,t)$

SL - the Stochastic Lanchester model

T_D - rv, time duration of combat

$T_{D|i}$ - rv, time duration of combat given a win by i = A,B or a draw, i = D

t - time

t_f - time at which L battles terminate

t_i - time at which i^{th} renewal event occurs

$V[A(t)]$ - variance of the rv $A(t)$

$W_k(t)$ - rv, superposed interkilling times

X - rv, marksman's ift

X_i - rv, ift of each member of the i^{th} side, i = A or B

$X_k(t)$ - rv, a transformed version of $A_k(t)$, k = 1,2,...

x - a value of $X_k(t)$ or
- a value of X or
- a value of X_i

x_1 - a particular value of x

x_2 - portion of initial x force initially engaged in Springall model

$x(t)$ - the number on side A at time t, in the L model

$x_L(t)$ - solution to the standard L Square Law equations with ics (a_0,b_0)

$Y(t)$ - rv, marksman's backward recurrence time at time t

$Y_k(t)$ - a transformed version of $B_k(t)$, $k = 1,2,...$

y - a value of $Y_k(t)$ or
- a value of $Y(t)$

y_1 - a particular value of y

y_2 - portion of initial y force initially engaged in Springall model

$y(t)$ - the number on side B at time t, in the L model

$y_L(t)$ - solution to the standard L Square Law equations with ics (a_0,b_0)

Z - rv, embedded killing process in the GR marksman model

Δa - change in a

$\Delta_A(t) = m_A(t) - x(t)$

$\Delta_B(t) = m_B(t) - y(t)$

$\Delta_{i|i}(t) = \Delta_i(t)$ given a win by i, i = A,B

$\alpha = p_A/\mu_A \triangleq$ A's individual kill rate (attrition coefficient for B side)

$\beta = p_B/\mu_B \triangleq$ B's individual kill rate (attrition coefficient for A side)

γ - non-combat attrition coefficient, B side or
- Springall attrition coefficient

$\delta(x-a)$ - Dirac Delta function $= \infty$ at $x = a$
$= 0$, elsewhere

δ - a constant, or
- Springall attrition coefficient

ε - an arbitrary positive constant

ζ - mean of rv X

$\eta = y - y_1$

η_0 - an arbitrary positive constant < 1

θ - the probability that an A side target is not acquired by a B side firer

λ - the probability that a B side target is not acquired by an A side firer

$\mu = \zeta/p = E[Z]$

μ_i - mean value of the ift on side i for the SL or GR model, i = A or B

μ_k - k^{th} moment about the origin, k = 1,2,...

$\xi = x - x_1$

ξ_0 - an arbitrary positive constant < 1

ρ - non-combat attrition coefficient, A side

σ_k - standard deviation of the losses on the A side in the SL process at time of the k^{th} event

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$\xrightarrow{D}$ - converges in distribution to

$\xrightarrow{P}$ - converges in probability to

$\times$ - cartesian product

$\triangleq$ - defined as equal to

LIST OF EXPOSITORY FIGURES

PART ONE - THEORY

I. INTRODUCTION

Great emphasis is placed in this document on assumptions, results, and concepts as they relate to small-to-moderate size battles. The reason for this is that the basic assumptions of current combat models do not hold for large numbers of combatants. Terrain compartmentalization, weapon ranges, terrain obstacles, weather and many other factors cause large scale battles to be a number of simultaneous and/or sequential smaller scale engagements.

Much of the current effort in weapons systems analysis or in combat analysis relies on utilization of various deterministic Lanchester (L) models of combat. It is important to emphasize at once that the models discussed here are only concerned with the progress of the fire-fight once it has begun. All pre-combat decisions have been made and the battle goes forward until terminated by the action itself or by tactical decisions (e.g. withdraw, surrender, etc.). This means that optimization techniques which influence pre-combat tactical decisions or pre-determined decisions to terminate are excluded. Optimization employs Differential Game Theory, Dynamic Programming, Game Theory, Control Theory, certain mathematical programming procedures or possibly other optimization techniques and requires good fire-fight models to operate on.

The most critical element in fire-fight analyses is the assignment of the attrition coefficients. For example, the Lanchester Square Law differential equations (see reference [10]*) governing the solution are

$$x'(t) = -\beta y, \qquad y'(t) = -\alpha x,$$

where x and y are taken to be the average numbers remaining on the two sides (A and B) respectively and where α and β are the B and A sides' attrition coefficients (A and B individual kill rates) respectively. It is customary to use p_A/μ_A for α, where p_A is the individual single round kill probability on the A side and μ_A is the individual mean interfiring time on the A side. All individuals on each side are assumed to be identical. A similar set of assumptions and definitions go with the B side.

A. The Assumptions Implicit in Using Lanchester's Equations

The analyses in this paper are based on the following additional assumptions, which are explicated for the Square Law for the sake of specificity. The details of other models will vary, but the analysis is similar and the attrition coefficients are defined in a manner appropriate to the model being considered. These assumptions are the basis for using L models mentioned above.

(1) The true model which is to be approximated, is termed a General Renewal (GR) model, and is illustrated in Figure 1 below. The characteristics of this model are:

(a) There are initially, a_0 on the A side and b_0 on the B side.

(b) Every member of the A side picks a B opponent at random (all are

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* Numbers in brackets [ ] refer to the list of references starting on p. 40.

visible and in range) and fires with a fixed kill probability p_A on every round fired with a general interfiring time (ift), which is a random variable(rv) X_A with mean μ_A. (Because a general rv has memory some particular distribution of fire is required for the GR model to be formulated mathematically. Different distributions of fire will, in general, give different results. However, no matter what distribution of fire is assumed in the GR model, as long as all combatants have a target (even if all are firing at the same target for example) the GR model reduces to the SL model when X_A and X_B are assumed negative exponentially distributed (ned). Thus we see that for L and SL the restriction that all opponents are visible and in range can be reduced to simply that every firer can see and possibly hit at least one target at all times.) For the GR model, it must be specified not only how many opponents are targets but how the fire is distributed over targets. This makes for possibly many different GR models for every SL and corresponding L model. To our knowledge this has not been pointed out before.

(c) Each marksman fires until he is killed or makes a kill (at which point he immediately shifts to a new target picked at random and resumes firing). The ammunition supply is unlimited.

(d) All fire independently.

(e) Similar assumptions apply to the B side.

(f) The battle continues until one side is annihilated.

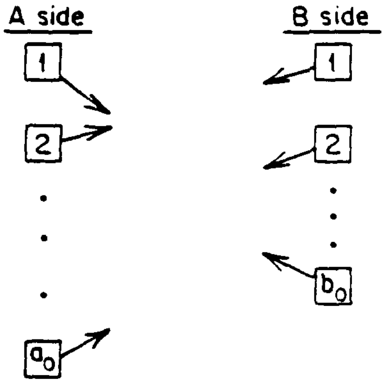

Figure 1. The GR Model

The derived quantities of interest are, $A(t)$ and $B(t)$, rvs which are the remaining numbers on each side with expected values $E[A(t)] = m_A(t)$ and $E[B(t)] = m_B(t)$. We note that these are marginal rvs and means from a joint distribution on some state-space.

This situation is a superposition of many terminating (non-classical) renewal processes (i.e. each marksman's firing process is a renewal process which terminates but, in general, not at one of his firing epochs). The reader is reminded that an ordinary general renewal process is one in which the times between events are independent, identically distributed (iid) rvs and that never terminates.

The only solutions to this model have been for the one-on-one duel (see reference [1]) and for the two-on-one duel (Gafarian and Ancker (1984)*).

(2) It is assumed that, if a_0 and b_0 are very large, $m_A(t)$ and $m_B(t)$ are well approximated by the mean value solutions to the "Stochastic Lanchester" (SL) equations which were first given a thorough treatment by Snow (1948). The SL model is exactly the GR model except that ifts are assumed to be negative exponentially distributed (ned). The ned assumption with its "no memory" property greatly simplifies the analysis but does not make it trivial.

(3) It is assumed that for large a_0 and b_0, the L model solution is a good approximation to $m_A(t)$ and $m_B(t)$ (the mean value time traces) of the SL model. Lanchester (see reference [10]) first proposed his model in 1916 and recognized that since it was deterministic and continuous in x,y and that the real process was stochastic and discrete, L should be considered an approximation of the stochastic process mean value.

(4) Finally, it is assumed that for large a_0 and b_0 the SL and GR models have negligible variance for much of the battle.

It is noted, parenthetically, that we consider all these models, (L, SL and GR) to be logically subsumed under the title "Lanchester" models. Thus SL and GR refine the original concepts of Lanchester in the direction of reality at the obvious cost of greater complexity.

It is this set of assumptions, (1) through (4) above, which are thoroughly examined, particularly the basic premises which purport to justify them. The main text (Part One) of this paper will examine all known theoretical results on this subject and in Part Two we collect, organize and reduce to common notation almost all known tabled results and curves of particular examples. These support the theoretical discussion.

The theoretical discussion will proceed in reverse order from assumption (3) backwards to (2) then take up (4) and close by considering other measures of effectiveness. First, several preliminary matters will be examined.

B. The Lanchester Square Law

It will be helpful to look at the solution to the Lanchester Square Law differential equations given above. These are the parameteric equations

$$x(t) = a_0 \cosh \sqrt{\alpha\beta}\, t - \sqrt{\beta/\alpha}\, b_0 \sinh \sqrt{\alpha\beta}\, t \ ,$$

$$y(t) = b_0 \cosh \sqrt{\alpha\beta}\, t - \sqrt{\alpha/\beta}\, a_0 \sinh \sqrt{\alpha\beta}\, t \ ,$$

and the phase-space equation (which is arrived at by dividing the second differential equation above into the first and solving),

$$y(t) = \{(\alpha/\beta)\ [x^2(t) - a_0^2] + b_0^2\}^{1/2}$$

all of which are only valid in the interval $(0,t_f)$ where,

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* References given by name(s) followed by a date are listed in the Annotated Bibliography beginning on p. 42. If the date is replaced by an asterisk (*), the material is new in this work.

$$t_f = (1/\sqrt{\alpha\beta})\tanh^{-1}(\sqrt{\alpha/\beta}\, a_0/b_0), \quad \text{if } \sqrt{\beta/\alpha}\, b_0/a_0 > 1, \quad \text{B wins, with } (b_0^2 - a_0^2\alpha/\beta)^{1/2} \text{ survivors}$$

$$= (1/\sqrt{\alpha\beta})\tanh^{-1}(\sqrt{\beta/\alpha}\, b_0/a_0), \quad \text{if } \sqrt{\beta/\alpha}\, b_0/a_0 < 1, \quad \text{A wins, with } (a_0^2 - b_0^2\beta/\alpha)^{1/2} \text{ survivors}$$

$$= \infty, \qquad \text{if } \sqrt{\beta/\alpha}\, b_0/a_0 = 1, \quad \text{which is a draw with no survivors.}$$

At time t_f, the combat has finished and one (or both) side(s) is annihilated. Beyond that time, the only interpretation which can be useful is to assume the states remain frozen at their t_f values. This is illustrated in Figure 2 below.

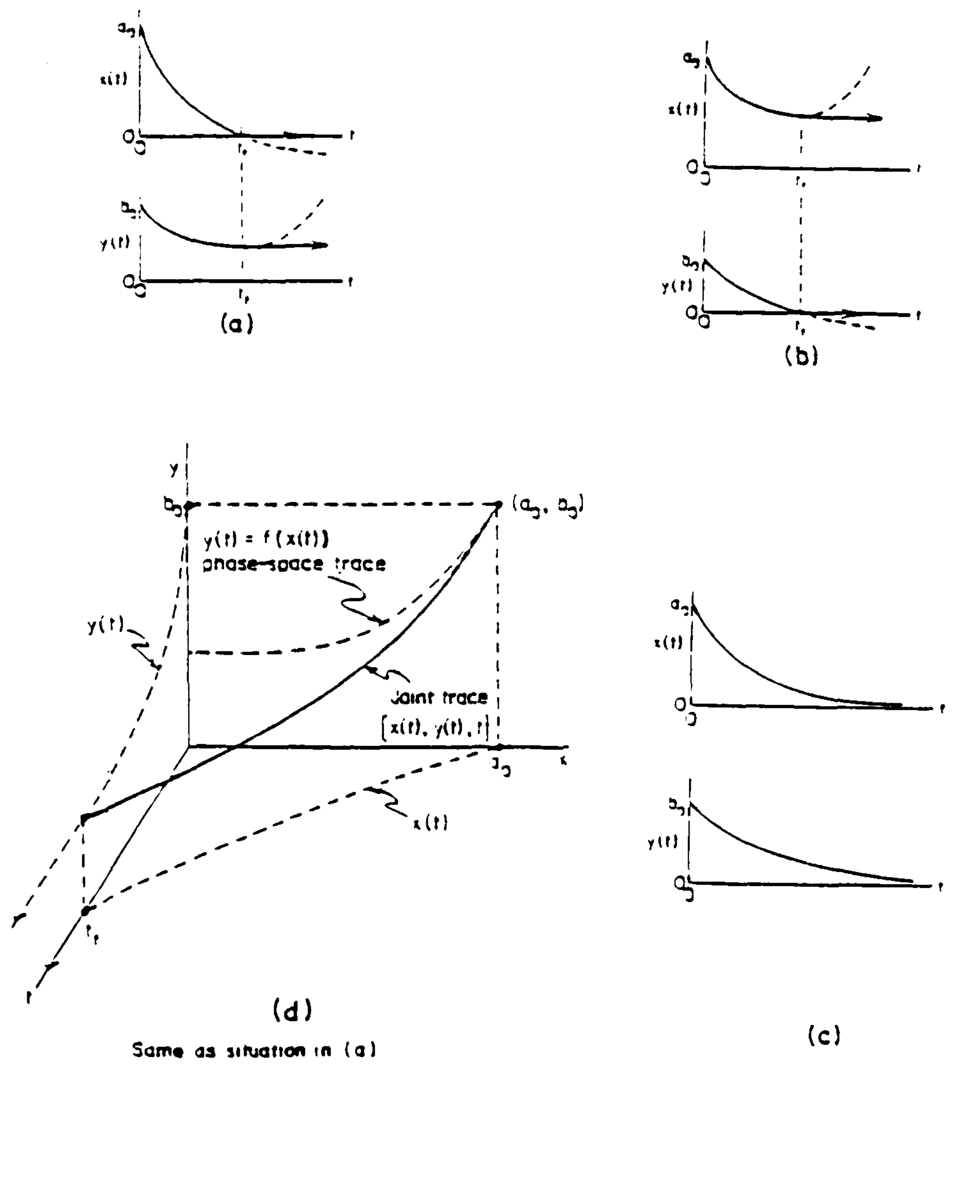


Figure 2. Typical Solutions to the Lanchester Square Law Equations

The important point to observe here is that at t_f the equations are no longer valid as they show an upturn on one side (impossible as there are no reinforcements) and at exactly the same time the other side starts to go negative (again an obvious impossibility). The only exception to this is when parity exists (i.e. $a_0 \sqrt{\alpha} = b_0 \sqrt{\beta}$); then t_f is infinite and the situation is as depicted in Figure 2(c).

C. Properties of the Equivalent Stochastic Solutions

It is useful to recall some properties of the joint probability mass function (pmf) of A(t) and B(t) (these apply equally well to the SL or GR models).

On the time-evolution joint probability state-space diagram (see Figure 3), the only possible states are at the lattice points (the intersections of the coordinate lines shown). At time t, the probability of being at a general lattice point (a,b) is designated by p(a,b,t). The states along the a-axis and the b-axis are absorption states (all opponents have been eliminated). All other states are transient. The state (0,0) can never be occupied. At time zero, the process is at (a_0,b_0) with probability one. For

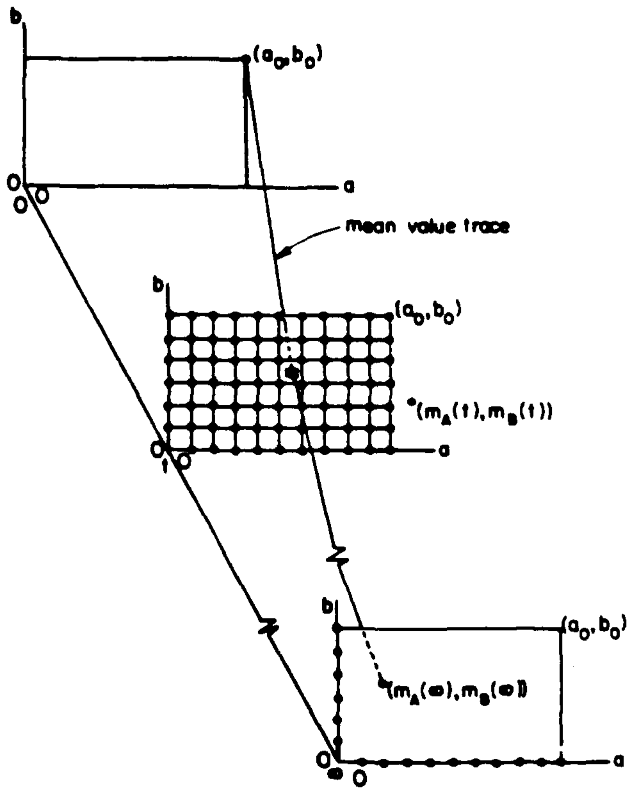


Figure 3. The Time-Evolution Joint Probability State-Space

all t > 0, all allowable states have non-zero probability of occurring. The point $(m_A(t),m_B(t))$ is where the mean value function pierces the state-space plane. In the limit as $t \to \infty$, all transient states have zero probability of occurring and all the probability mass is located on the a and b axes. This is still a joint pmf but, of course, all probabilities are just on the two axes as shown. The probability the A side has won by time t is just the sum of all the probabilities on the a-axis, and similarly for the B side. The total win probabilities are these functions evaluated at $t = \infty$.

There seems to be some confusion in the literature about the joint pmfs shown, the marginal pmfs and the conditional absorption pmfs. In Figure 3 (at any time, t) the marginal pmf for A is simply all the probability mass projected onto the a-axis and similarly for B. Given that A has won, the conditional pmf is found by dividing each mass point on the a-axis by the sum of the mass points on the a-axis and, similarly for B. Particular confusion appears to occur in connection with moments of the three possible pmfs (joint, marginal and conditional) especially in the terminal state ($t = \infty$) and it is important to keep clearly in mind which is intended.

On the time-independent joint probability state-space (see Figure 4) the diagonals labelled $k = 1, 2, \ldots, (a_0+b_0-1)$ contain the possible states after the first, second, third, etc. events have occurred. The probability masses located on each diagonal ($k \leq \min(a_0,b_0)$) form a proper pmf and add to one. For $k > \min(a_0,b_0)$, all absorption probabilities associated with smaller ks must be included with those on the diagonal to form a proper pmf (see e.g. the broken line for $k = a_0$ in Figure 4.) A random walk may be visualized on this space with each step either to the left or downward and terminating on the a or b-axis.

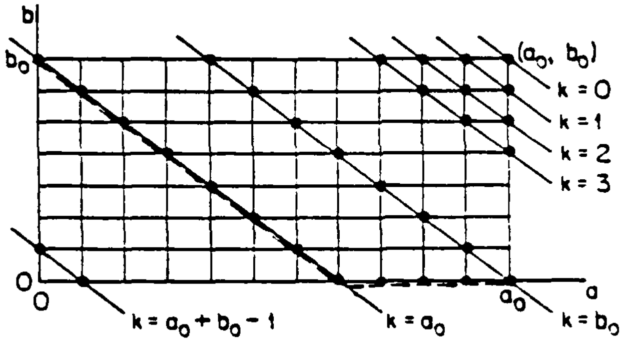


Figure 4. The Time-Independent Joint Probability State-Space

It is important to note here that the concept of "breakpoint" (i.e., defeat before annihilation) complicates the mathematics but does not change the conclusions of the following analyses in any material way. Breakpoints create absorbing barriers at specific positive values of a and b (a_f less than a_0 and b_f less than b_0). This means that absorption probabilities become significantly large earlier in the battle and their role in the differences between L and SL measures of battle outcomes is effective earlier.

We now proceed to examine important theoretical points about the assumptions outlined above.

II. THE ERROR IN CONSIDERING LANCHESTER'S SOLUTIONS AS AN APPROXIMATION TO STOCHASTIC LANCHESTER MEAN VALUE FUNCTIONS

Before proceeding with details, it is noted that essentially the question here is - how well does the L joint trace (e.g., Figure 2(d)) approximate the corresponding SL mean value trace of Figure 3 (or equivalently their marginal or phase-space projections)?

In an important paper not readily accessible and thus largely unnoticed Hardeck and Hilden (1967, equations (10), p. 5) have shown, when their result is expressed in terms of the number of survivors, that

$$\left.\begin{aligned} m_A'(t) &= -\sum_{a=a_f+1}^{a_0}\ \sum_{b=b_f+1}^{b_0} r_B(a,b,t)\ p(a,b,t) = -E[r_B(A(t),B(t),t)] \\ m_B'(t) &= -\sum_{a=a_f+1}^{a_0}\ \sum_{b=b_f+1}^{b_0} r_A(a,b,t)\ p(a,b,t) = -E[r_A(A(t),B(t),t)] \end{aligned}\right\} \qquad (1)$$

for a general SL process (with breakpoints) and where kill rates (r_A, r_B) also depend on α and β even though this dependency is not explicitly shown. Later, for the same general SL process with kill rates not a function of time and where the combat goes to annihilation, Clark (1969 equations (73), (74) and (75), p. 79) has derived these same expressions except, of course, that the kill rates were not dependent on time.

Although neither Hardeck and Hilden nor Clark say so, these expressions conclusively prove that compared to any equivalent L formulation, the SL form will be different.

This comes about because the right hand side (rhs) of equation (1) cannot be made to look like the rhs of equivalent L equations. There are two reasons this cannot be done. They are:

(1) There are no absorption probabilities utilized in the SL formulation for computing the expected kill rates because kill rates are, by definition, zero as soon as one side is annihilated (or reaches its breakpoint).

(2) A(t) and B(t) are correlated.

In any possible L model one or the other or both of these facts will create a difference between the L differential equations and the equivalent SL mean value differential equations.

This will be illustrated below by all particular examples in the literature known to the authors.

A. The Square Law

1. Time-Independent Kill Rates

The first analytic attention to this problem was given by Snow (1948). However, he first gives a more general L problem (see equations (3) and (25) in Snow); namely

$$\left.\begin{aligned} x'(t) &= -\rho x - \beta y + g_A(t), \\ y'(t) &= -\gamma y - \alpha x + g_B(t), \end{aligned}\right\} \tag{2}$$

and then derives (see p. 24, Snow) the more general SL equivalent

$$\left.\begin{aligned} m_A'(t) &= -\rho m_A - \beta m_B + g_A(t) + \beta \sum_{b=1}^{b_0} b p(0,b,t), \\ m_B'(t) &= -\gamma m_B - \alpha m_A + g_B(t) + \alpha \sum_{a=1}^{a_0} a p(a,0,t) , \end{aligned}\right\} \tag{3}$$

where ρ and γ are non-combat attrition coefficients and $g_A(t)$ and $g_B(t)$ are reinforcement functions.

It should be observed, in passing, that Tompkins (1953, p. 37, equation 3.2) rederived these results (with $g_A(t) = g_B(t) \equiv 0$), apparently being unaware of Snow's priority.

The Square Law equations are easily derived by setting $\rho = \gamma = g_A(t) = g_B(t) \equiv 0$ in equations (2) and (3). Note that the L differential equations differ from the SL differential equations by the terms containing absorption probabilities. The SL Square Law is also easily derived from equation (1) by setting $r_A(a,b,t) = r_A(a,b) = \alpha a$ and $r_B(a,b,t) = r_B(a,b) = \beta b$.

We note from the general differential-difference state equations (Clark 1969, equations (51) through (57) pp. 69,70) with Square Law rates inserted that the state probabilities have the following properties (see Figure 5):

(1) The absorption probabilities ($p(a,0,t)$, $p(b,0,t)$) shown in Figure 5(a) are monotonically increasing functions with value zero and slope zero at $t = 0$ and asymptotes $p(a,0,\infty)$ and $p(0,b,\infty)$. The curvature is initially positive and changes to negative.

(2) The initial state probability, $p(a_0,b_0,t)$, is a monotonically decreasing function with value one and with slope $-(a_0\alpha + b_0\beta)$ at $t = 0$ and is asymptotic to zero at infinity. The curvature is always positive (see Figure 5(b)).

(3) All other state probabilities have values zero and slope zero at time zero and are asymptotic to zero at infinity. The curvature is initially positive, changes to negative then goes back to positive (see Figure 5(c)).

These facts along with the equations for the mean value functions for the Square Law show that $m_A(t)$ and $m_B(t)$ have values a_0, b_0 and slopes $-\beta b_0$ and $-\alpha a_0$, at $t = 0$ and are monotonically decreasing to positive asymptotes with slopes zero at infinity. They also have positive curvature everywhere tending to zero at infinity.

The Square Law L equations have the same initial values and slopes as the SL mean value functions and have positive curvature (but different from SL) for positive $t < t_f$. It should be noted that in comparing all L and SL pairs

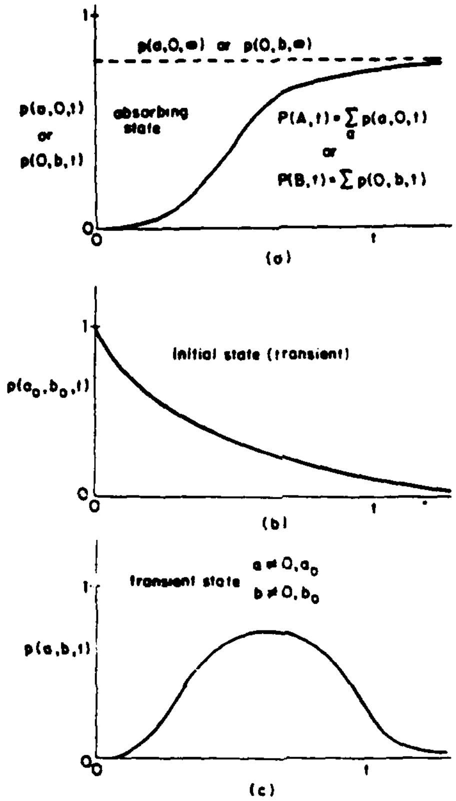


Figure 5. Characteristics of SL Square Law State Probabilities

of equations, x is always identified with m_A and y with m_B. This is the only rational interpretation of x and y and has always been so recognized by most analysts including Lanchester himself as mentioned earlier. Also it should be observed that $m_A(t)$, $m_B(t)$, $x(t)$ and $y(t)$ will be written as m_A, m_B, x, and y on the rhs of all equations for brevity, always keeping in mind that they are functions of time.

With these conventions in mind, it is clear that equations (2) and (3) differ only in that the rhs of the SL equations contain terms proportional to the average number of survivors. Clark (1969) has dubbed these the "bias" terms of the SL equations. This terminology is dropped in this paper because bias has a well known, rigorous meaning in statistics which does not apply here as this paper deals with purely probabilistic models with no question of statistical sampling involved. The focus here is on the difference in the solutions to the differential equations (i.e. the difference between the L functions and SL mean value functions).

The principal point to note here is that for $t > 0$

$$1 > p(a,0,t) > 0 \quad , \quad 0 < a \le a_0 \; ,$$
$$1 > p(0,b,t) > 0 \quad , \quad 0 < b \le b_0 \; ,$$

and these probabilities are monotonically increasing (their derivatives are positive for all t). Thus, although $m_A(t)$ and $x(t)$, and $m_B(t)$ and $y(t)$ start at the same point at $t = 0$ (their initial conditions require $m_A(0) = x(0) = a_0$ and $m_B(0) = y(0) = b_0$) in general, the L and SL mean value trajectories will differ for all t (the only exception is that they may cross, and therefore be equal, at certain specific times).

2. Time-Dependent Kill Rates

Hardeck and Hilden (1967, p. 8) have shown that for time-dependent kill probabilities $p_A(t)$ and $p_B(t)$ and firing rates $r_A(t)$ and $r_B(t)$ that

$$m_A'(t) = -p_B(t)r_B(t)\left[m_B - \sum_{b=1}^{b_0} b\; p(0,b,t)\right] ,$$

$$m_B'(t) = -p_A(t)r_A(t)\left[m_A - \sum_{a=1}^{a_0} a\cdot p(a,0,t)\right] .$$

The corresponding L equations again only differ by the terms containing the absorption probabilities, and the comments in section A.1. above apply.

This is fundamentally different than the other processes that are examined here in that the killing process embedded in the firing process is nonhomogeneous Poisson and is therefore not renewal (i.e., it is not iid).

B. The Square Law with Breakpoints

Craig (1975 equations (101), and (102) pp. 160 and 161) has shown

$$\left.\begin{aligned} m_A'(t) &= -\beta m_B + \beta\left\{\sum_{b=b_f+1}^{b_0} bp(a_f,b,t) + b_f \sum_{a=a_f+1}^{a_0} p(a,b_f,t)\right\}, \\ m_B'(t) &= -\alpha m_A + \alpha\left\{\sum_{a=a_f+1}^{a_0} ap(a,b_f,t) + a_f \sum_{b=b_f+1}^{b_0} p(a_f,b,t)\right\}. \end{aligned}\right\} \quad (4)$$

The corresponding L equations are the same as the annihilation case except that a_f and b_f occur in the boundary conditions in an obvious way.

Although the sums on the rhs of equations (4) (which produce the L-SL differences) are more complicated than for equations (3), the conclusions are not altered materially. Again we get the simple Square Law by letting $a_f = b_f = 0$.

C. The Linear Law

For the well known L Linear Law given by,

$$x'(t) = -\beta xy, \qquad y'(t) = -\alpha xy, \qquad (5)$$

the parametric (time) solutions are, for $\alpha a_0 \neq \beta b_0$,

$$x(t) = \frac{(\beta b_0 - \alpha a_0)a_0}{\beta b_0 \exp[(\beta b_0 - \alpha a_0)t] - \alpha a_0} ,$$

$$y(t) = \frac{(\alpha a_0 - \beta b_0)b_0}{\alpha a_0 \exp[(\alpha a_0 - \beta b_0)t] - \beta b_0} ,$$

and when $\alpha a_0 = \beta b_0$, $x(t) = a_0/(\alpha a_0 t + 1)$, $y(t) = b_0/(\beta b_0 t + 1)$. The phase-space solution is

$$y = (\alpha/\beta)x + (\beta b_0 - \alpha a_0)/\beta.$$

All these solutions are valid in the interval $(0,t_f)$ where $t_f = \infty$. If

$\alpha a_0 > \beta b_0$, A wins, $x(\infty) = (\alpha a_0 - \beta b_0)/\alpha$, $y(\infty) = 0$,

$\alpha a_0 < \beta b_0$, B wins, $x(\infty) = 0$, $y(\infty) = (\beta b_0 - \alpha a_0)/\beta$,

$\alpha a_0 = \beta b_0$, draw with no survivors, $x(\infty) = y(\infty) = 0$.

Clark (1969 equations (80) and (81) on p. 81) has shown that the equivalent SL differential equations are

$$\left.\begin{aligned} m_A'(t) &= -\beta E[A(t)B(t)] = -\beta m_A m_B - \beta \operatorname{cov}[A(t)B(t)], \\ m_B'(t) &= -\alpha E[A(t)B(t)] = -\alpha m_A m_B - \alpha \operatorname{cov}[A(t)B(t)] \quad . \end{aligned}\right\} \qquad (6)$$

It should be noted here that the phase-space solution for equations (5) derived by dividing the first equation by the second equation and solving are exactly the same as obtained by the same process in equations (6). Other versions of the Linear Law will give the same phase-space equations, but different time traces. In general, replace xy on the rhs of equations (5) by any general function $g(x,y)$ and a type of Linear Law will result. The fact that the L and SL time-independent phase-space equations are exactly the same (a straight line starting at $x(0) = a_0$, $y(0) = b_0$ and $m_A(0) = a_0$, $m_B(0) = b_0$ and with the same slopes) does not mean the lines are identical. The L lines terminate either on one of the axes or the origin, the SL lines terminate at $(m_A(\infty),m_B(\infty))$ which are always positive (see Figure (6a)). Unless otherwise stated when "Linear Law" is used it will mean the versions given in equations (5) and (6).

Clark did not actually show the second version of the rhs of equations (6), but they are obvious from basic probability theory. Again the L and the SL mean value differential equations differ, this time by the covariance terms. Thus the SL version could only be the same as the L version if A(t) and B(t) were independent or uncorrelated. Since all the p(a,b,t) functions with $a,b \neq 0$ are known (Clark (1969), p. 102 equation (106)) and are not of the form p(a,t) p(b,t) (i.e., product of the marginal probabilities), then A(t) and B(t) are not independent and therefore A(t) and B(t) may, but, in general, will not have zero covariance. Thus although the

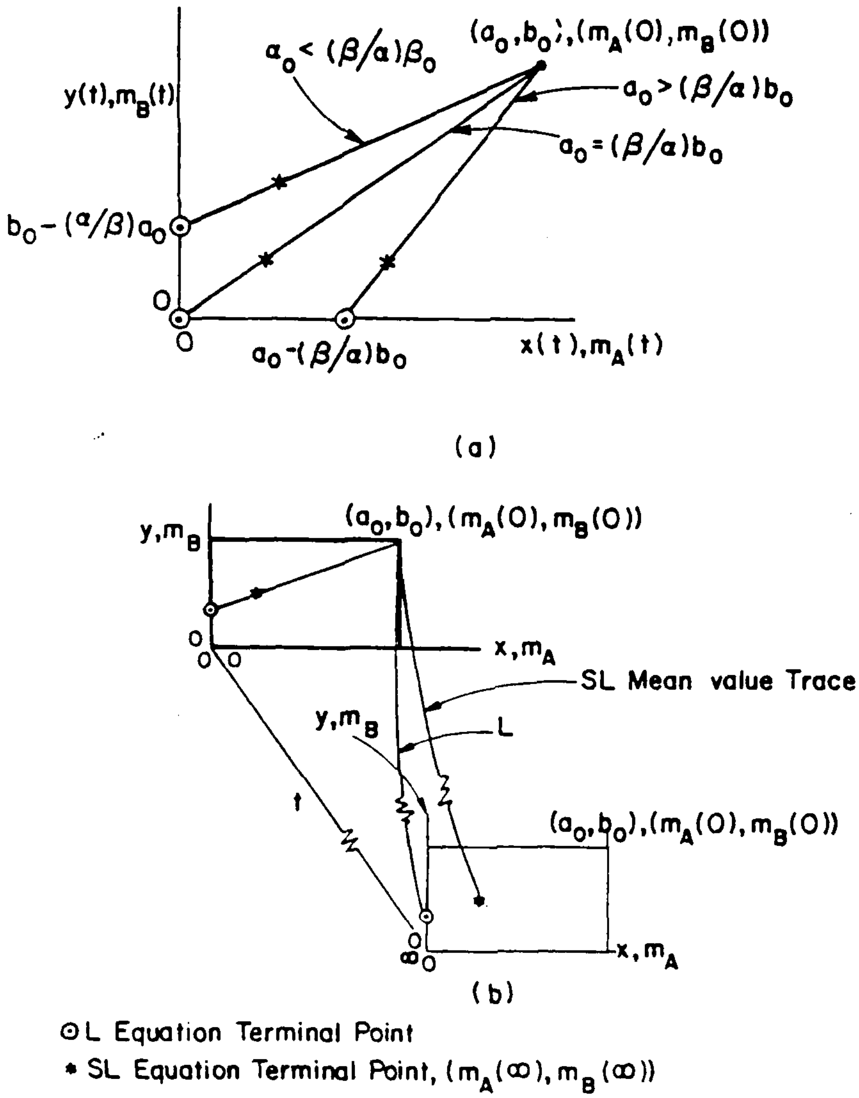


Figure 6. Comparison of L and SL Solutions for the Linear Law.

phase-space equations are collinear, the mean value time-traces for L and SL differ (see Figure 6b).

Equation (6) may easily be derived from equations (1) by letting $r_A(a,b,t) = r_A(a,b) = \alpha ab$ and $r_B(a,b,t) = r_B(a,b) = \beta ab$.

D. Two Special Models

(1) Clark (1969, p. 151) has investigated a special case where acquisition probabilities are involved. The L equations are

$$\left.\begin{aligned} x'(t) &= -\beta y(1-\theta^{x}) = -\beta y + \beta y \theta^{x} , \\ y'(t) &= -\alpha x(1-\lambda^{y}) = -\alpha x + \alpha x \lambda^{y} , \end{aligned}\right\} \tag{7}$$

and the equivalent SL equations are

$$\left.\begin{aligned} m_A'(t) &= -\beta E[B(t)(1-\theta)^{A(t)}] = -\beta m_B + \beta m_B E[\theta^{A(t)}] \\ &\qquad + \beta \operatorname{cov}[B(t)\theta^{A(t)}] , \\ m_B'(t) &= -\alpha E[A(t)(1-\lambda^{B(t)}] = -\alpha m_A + \alpha m_B E[\lambda^{B(t)}] \\ &\qquad + \alpha \operatorname{cov}[A(t)\lambda^{B(t)}] , \end{aligned}\right\} \tag{8}$$

where $0 < \theta, \lambda < 1$ are target nonacquisition probabilities for A and B respectively.

Again, there is a difference in the two versions which is even more pronounced as $E[\theta^{A}]$ cannot yield θ^{m_A}. We note that equation (8) may be obtained from equations (1) by letting $r_A = \alpha a(1-\lambda^{b})$ and $r_B = \beta b(1-\theta^{a})$.

(2) Springall (1968) has thoroughly investigated a rather complicated model whose L formulation is given by,

$$x'(t) = -\beta xy - \delta y,$$

$$y'(t) = -\alpha xy - \gamma y,$$

where δ and γ are additional fixed attrition coefficients. The initial numbers engaged are x_2 and y_2, which are fractions of the initially available forces (a_0, b_0 respectively). The remaining forces are in reserve and are deployed one by one as the initally engaged forces are reduced in such a manner as to keep the engaged forces at levels x_2 and y_2 respectively, until all reserves are committed and then the battle proceeds either to annihilation or to a specified breakpoint.

The corresponding SL mean value functions are too complicated to be reproduced here but examples of the outcome of such battles are given in Part Two and show substantial L function - SL mean value function differences.

E. An Analysis of the L-SL Mean Value Function Difference

Define the L-SL mean value function differences to be

$$\left.\begin{aligned} \Delta_A(t) &= m_A - x , \\ \Delta_B(t) &= m_B - y . \end{aligned}\right\} \tag{9}$$

1. The Square Law

Craig (1975, p. 68) has shown that for the SL Square Law

$$\left.\begin{aligned}
\Delta_A(t) &= \int_0^t \{S_B(\tau)\cosh\sqrt{\alpha\beta}\,(t-\tau) - \sqrt{\beta/\alpha}\,S_A(\tau)\sinh\sqrt{\alpha\beta}(t-\tau)\}d\tau\\
\Delta_B(t) &= \int_0^t \{S_A(\tau)\cosh\sqrt{\alpha\beta}\,(t-\tau) - \sqrt{\alpha/\beta}\,S_B(\tau)\sinh\sqrt{\alpha\beta}\,(t-\tau)\}d\tau\\
&\text{where } 0 < t \leq t_f, \text{ and}\\
S_A(t) &= \alpha\sum_{a=1}^{a_0} ap(a,0,t),\\
S_B(t) &= \beta\sum_{b=1}^{b_0} bp(0,b,t).
\end{aligned}\right\} \quad (10)$$

$S_A(t)$ and $S_B(t)$ are the terms in the SL differential-difference equations for the ordinary Square Law formulation causing L-SL differences. Strictly speaking, Craig uses the breakpoint model for equations (9) and (10) but this includes the annihilation model we are considering. Craig investigated (10) by using various particular values of a_0, b_0, α, β. We shall go a bit further here to draw some more general conclusions.

Equations (10) may easily be rewritten as

$$\left.\begin{aligned}
\Delta_A(t) &= \{\int_0^t [S_B(\tau) - \sqrt{\beta/\alpha}\,S_A(\tau)]\exp[\sqrt{\alpha\beta}\,(t-\tau)]d\tau\\
&\quad + \int_0^t [S_B(\tau) + \sqrt{\beta/\alpha}\,S_A(\tau)]\exp[-\sqrt{\alpha\beta}\,(t-\tau)]d\tau\}/2,\\
\Delta_B(t) &= \sqrt{\alpha/\beta}\,\{-\int_0^t [S_B(\tau) - \sqrt{\beta/\alpha}\,S_A(\tau)]\exp[\sqrt{\alpha\beta}\,(t-\tau)]d\tau\\
&\quad + \int_0^t [S_B(\tau) + \sqrt{\beta/\alpha}\,S_A(\tau)]\exp[-\sqrt{\alpha\beta}\,(t-\tau)]d\tau\}/2.
\end{aligned}\right\} \quad (11)$$

Now, by the Second Mean Value Theorem equation (11) can be written as

$$\left.\begin{aligned}
\Delta_A(t) &= \{1-\exp(-\sqrt{\alpha\beta}t)\}\{I_1(c,t) + I_2(d,t)\}/2\sqrt{\alpha\beta},\\
\Delta_B(t) &= \{1-\exp(-\sqrt{\alpha\beta t})\}\{-I_1(c,t) + I_2(d,t)\}/2\beta, \quad \text{where}\\
I_1(c,t) &= [S_B(c) - \sqrt{\beta/\alpha}\,S_A(c)]\exp(\sqrt{\alpha\beta}t),\\
I_2(d,t) &= S_B(d) + \sqrt{\beta/\alpha}\,S_A(d), \quad \text{and}\\
0 &< c,d < t \leq t_f.
\end{aligned}\right\} \quad (12)$$

$p(a,0,t)$ and $p(b,0,t)$ are positive, monotonically increasing functions of t, as previously mentioned, and thus $S_A(t)$ and $S_B(t)$ are also. It is noted that although $S_A(t)$ and $S_B(t)$ are functions of α and β, they are absolutely bounded for any $t > 0$. This comes about from the fact that $\sum_{a=1}^{a_0} p(a,0,\infty) = P(A)$, the probability A wins, and $\sum_{b=1}^{b_0} p(0,b,\infty) = P(B)$, the probability B wins. Therefore, certainly $S_A(t) < \alpha\ a_0 P(A)$ and $S_B(t) < \beta\ b_0 P(B)$ where $P(A) + P(B) = 1$. Thus, by inspection of equations (12) and noting that $I_2(d,t) > 0$ for $t > 0$,

(1) If either Δ_A or Δ_B is negative at any time, t, (or any range of times) then the other must be positive.

(2) If $\Delta_A(t)$ or $\Delta_B(t)$ is zero for some particular values of α,β,a_0,b_0, and t, then the other must be positive at these points. Such points are crossing points for the Δ with the zero value.

(3) Both differences may be positive but from (1) above they cannot both be negative at the same time. In particular they are both positive for strict L parity (i.e. $a_0 = b_0$, $\alpha = \beta$).

And from many particular examples in Part Two-I, we have

(4) Δ_A or Δ_B may be negative, positive or zero and these differences can be as high as about 30 or 40 percent of the initial values.

Furthermore,

(5) Although all the above applies for $t \leq t_f$, in fact, it can be shown they are also true for true $t > t_f$.

(6) Finally, from (1) thru (5) above, contrary to some statements in the literature (for example, see Farrell (1976, p. 5)), the L equations cannot be used as universal bounds on the SL mean value functions.

2. The Linear Law

Taylor (1983, equation 4.12.24 p. 505) has shown for the Linear Law that it is easy to derive the relation $\Delta_A(t) = (\beta/\alpha)\Delta_B(t)$ which clearly indicates that, in this case, the differences are always of the same sign and a crossover in one is accompanied by a crossover in the other at exactly the same time.

F. The Difference Near Time Zero for Certain Special Cases

Many authors have noted that for the Square Law (even with variations such as break-points or reinforcements) the L and SL mean value differential equations are identical if absorption probabilities are taken to be neligible, that is set them equal to zero. This only makes sense for very large initial numbers and for early in the combat. Snow (1948) first mentioned this and it was later exploited by Marshall (1965), Clark (1969) and Koopman (1970).

Grainger (1976, Appendix G, pp. 87 and 88) has shown that for L models given by

$$x'(t) = -\beta x^c y^d, \qquad y'(t) = -\alpha x^e y^f, \tag{13}$$

where c,d,e,f are positive integers or zero, the corresponding SL mean value functions (near enough to t=0 so that the absorption probabilities are negligible) are given by

$$m_A'(t) = -\beta E[A^c B^d], \qquad m_B'(t) = -\alpha E[A^e B^f]. \qquad (14)$$

In general, A(t) and B(t) are not independent, and therefore functions of A(t) and B(t) are not independent and the only values of c,d,e, and f for which (13) and (14) coincide (near t=0, of course) are (c=0, d=1), (c=1, d=0) and (c=0,d=0) for the first equation along with (e=0, f=1), (e=1,f=0) and (e=0,f=0) for the second equation. This gives nine possible combinations of which only (a) (c=0,d=1,e=1,f=0), the standard Square Law and (b) (c=d=e=f=0), the Linear Law model where the battle is a sequence of one-on-one duels, are interesting situations. For these, the difference is nearly zero in the neighborhood of t=0. For the general Linear Law case (and other situations) this indicates that there may be large differences even near time zero. Snow (1948, p. 25) earlier came to the same conclusion for the Square Law.

G. Summary

Summarizing the major points in this section:

(1) All L-SL mean value equivalent pairs differ for all times except at possible crossing points. These differences may be very large.

(2) At least for the Square Law, the L trajectories are neither a universal upper or lower bound of the SL mean value trajectories.

(3) Even near time zero, the L and SL mean value trajectories may differ considerably (they do not differ materially for the Square Law and the sequence of one-on-one duels version of the Linear Law).

III. FALLACIES IN CONTINUOUS STATE-SPACE APPROXIMATIONS FOR THE SL MEAN VALUE FUNCTION

There have been several attempts to show that specific SL models converge in probability (in some sense) to L equivalents. This is another effort to show that L is a good approximation to SL for large numbers. These attempts have been widely misinterpreted and generally misunderstood. It should be understood that our analyses here reinforce the results in Section II above.

A. Rigorous Convergence in Probability

Etter (1971) and Karr (1976) have shown rigorously that transformed versions of the SL Linear and Square Laws converge in probability to the L laws. Karr also shows that transformed Mixed (one side Linear, one side Square) and Square Law versions with continuous reinforcement also converge in probability to the corresponding L laws. Their proofs differ (Karr uses probability arguments and Etter relies on function theory) but the transformations are essentially the same. In both, a state space with initial conditions (ics) of (a_0,b_0) is expanded in discrete jumps (keeping the ratio of ics constant) to ics $(2a_0,2b_0),(3a_0,3b_0),\ldots,(ka_0, kb_0)$. In Karr, the rvs $A_k(t)$, $B_k(t)$, where the subscript k refers to the k^{th} space in the sequence above, are then transformed by

$$X_k(t) = A_k(t)/k, \qquad Y_k(t) = B_k(t)/k. \tag{15}$$

This takes the expanded space with lattice points defined on all non-negative integer pairs (a,b), except (0,0), such that $a \le ka_0$, $b \le kb_0$ and transforms it to a new (X_k, Y_k) space with ics (a_0, b_0) and exactly the same number of lattice points as the untransformed space but now they are spaced $1/k$ units apart in both the x and y directions (see Figure 7). Of course, the mapping in (15) brings exactly the probabilities in the untransformed space onto the corresponding lattice points in the transformed space.

It is very important to observe that in any transformation of the type given by equation (15), if a comparison is to be made with the corresponding Lanchester equations the same transformations must be made on the L equations (see Figure 7, in which x_L and y_L are the standard L solutions).

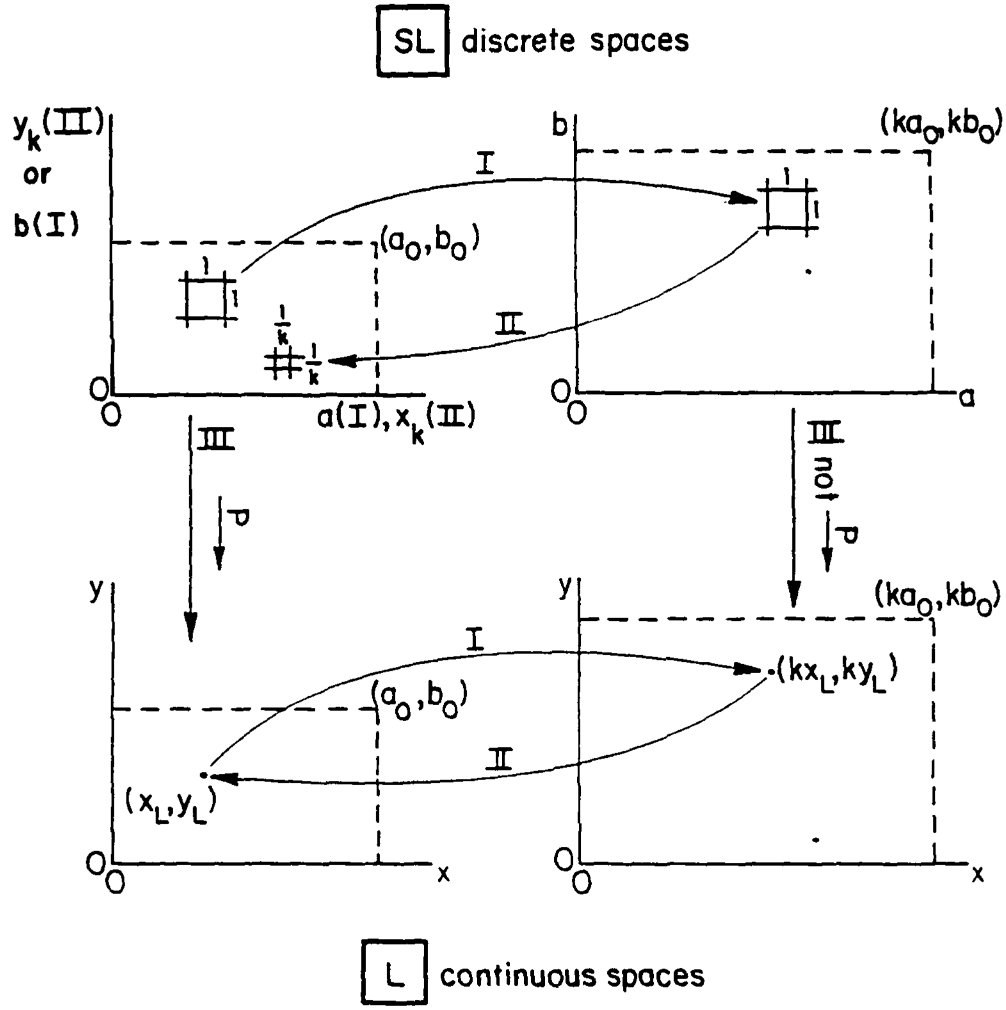
Etter's transformation replaces equation (15) by $X_k(t) = A_k(t)\xi_0/k$, $Y_k(t) = B_k(t)\eta_0/k$, where ξ_0, η_0 are positive constants ≤ 1. He also implies (correctly) that this transformation is similar to $X_k(t) = A_k(t)\xi_0/k(a_0 + b_0)$, $Y_k(t) = B_k(t)\eta_0/k(a_0 + b_0)$. The corresponding ics on these two differently transformed spaces are $(a_0\xi_0, b_0\eta_0)$ and $(a_0\xi_0/(a_0 + b_0), b_0\eta_0/(a_0 + b_0))$ respectively and the lattice cell sizes are ξ_0/k by η_0/k and $\xi_0/k(a_0 + b_0)$ by $\eta_0/k\,(a_0 + b_0)$, respectively.

There are no essential differences in all these transformations as they all require that the ics on the untransformed spaces go to infinity at the <u>same rate</u> and that the cells in the transformed spaces retain their shapes (the <u>ratio</u> of the sides are constant) and decrease uniformly. Thus as $k \to \infty$ the untransformed space increases without limit and the transformed space remains the same size and shape but the number of lattice points increases with decreasing distances between them, (see Figure 7).

It should be noted that for some of Karr's results, he also dilates time by t/k.

To clarify the situation let us use the Karr transformation directly on the Square Law SL state equations. To the best knowledge of the authors this has not been done in this manner before. First we define for a_0, b_0 and $k = 1,2,\ldots$

$$\left.\begin{aligned} p_k(a,b,t) &= P\{A_k(t) = a,\ B_k(t) = b,\ ic(ka_0, kb_0)\}, \\ &= 0,\ a > ka_0 \text{ or } b > kb_0\ , \\ &= 0,\ a < 0 \text{ or } b < 0\ , \\ &= 0,\ a = 0 \text{ and } b = 0\ , \\ &= 0,\ t < 0\ , \\ &= 1,\ t = 0,\ a = ka_0,\ b = kb_0\ . \end{aligned}\right\} \tag{16}$$



I - expanding the state space
II - transformation (equation (15))
III - limiting situations as $k \to \infty$

Figure 7. Convergence in Probability for Every Fixed Time, t

Thus,

$$\frac{\partial p_k(a,b,t)}{\partial t} = \alpha a[p_k(a,b+1,t) - p_k(a,b,t)] + \beta b[p_k(a+1,b,t) - p_k(a,b,t)] \tag{17}$$

with ic $p_k(ka_0,kb_0,0) = 1$ (see Clark (1969, p. 69)). Note that for the limiting process (where $k \to \infty$) only interior points need be considered and boundary equations are ignored. Using transformation equations (15), which imply that $x = a/k$, $y = b/k$ (where x and y are values of $X_k(t)$ and $Y_k(t)$ respectively), and the mapping is on the probabilities only (i.e. $p_k(a,b,t) = p_k(x,y,t)$),

$$\frac{\partial p_k(x,y,t)}{\partial t} = \frac{\alpha x[p_k(x,y+1/k,t) - p_k(x,y,t)]}{1/k} + \frac{\beta y[p_k(x+1/k,y,t) - p_k(x,y,t)]}{1/k} \tag{18}$$

with ic $p_k(a_0,b_0,0) = 1$.

The probabilities in equation (18) are on all the lattice points x,y in the transformed space, where the lattice points are rational number pairs on the rectangle $\{[0,a_0]\times[0,b_0]\}$. However, consider <u>any fixed point</u> (x_1,y_1), rational or irrational, in the rectangle. Each is in <u>some</u> cell bounded by the lattice points for every k (see Figure 8 below). The cell sizes are diminishing as k increases. By replacing

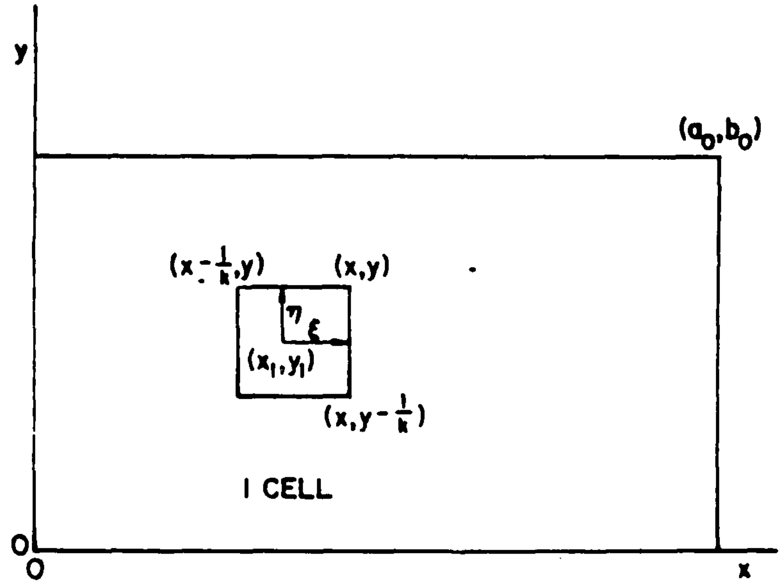


Figure 8. Transformed SL State-Space at Each t.

x by $x_1 + \xi$, y by $y_1 + \eta$, $x + 1/k$ by $x_1 + \xi + 1/k$, and $y + 1/k$ by $y_1 + \eta + 1/k$ equation (18) becomes

$$\frac{\partial p_k(x_1+\xi,y_1+\eta,t)}{\partial t} = \frac{\alpha(x_1+\xi)[p_k(x_1+\xi,y_1+\eta+1/k,t) - p_k(x_1+\xi,y_1+\eta,t)]}{1/k} + \frac{\beta(y_1+\eta)[p_k(x_1+\xi+1/k,\ y_1+\eta,t) - p_k(x_1+\xi,\ y_1+\eta,t)]}{1/k} \tag{19}$$

with ic, $p_k(a_0,b_0,0) = 1$. Note that $0 \le \eta,\xi \le 1/k$, therefore as $k \to \infty$, $\eta,\xi \to 0$.

Now, essentially, Karr and Etter have shown that there exists a continuous function, p, with first derivatives defined everywhere on the transformed rectangular state space which p_k tends to in the limit as $k \to \infty$. Taking the limit as $k \to \infty$ on both sides of (19) and dropping the subscripts on x and y yields

$$\frac{\partial p(x,y,t)}{\partial t} = \alpha x \frac{\partial p(x,y,t)}{\partial y} + \beta y \frac{\partial p(x,y,t)}{\partial x} . \tag{20}$$

In the limiting process, the p functions have gone from a joint probability mass function on the lattice points to a joint probability density function (pdf) on the transformed continuous rectangular state-space. Thus, the initial condition is given by,

$$p(a_0,b_0,0) = \delta(x - a_0)\,\delta(y - b_0). \tag{21}$$

Williams (1963, p. 38), Koopman (1970, p. 870) and Taylor (1972, p. I-44) have shown that, using the method of characteristics

$$\frac{dx}{\beta y} = \frac{dy}{\alpha x} = \frac{dt}{-1} = \frac{dp}{0} ,$$

the solution is

$$\left.\begin{aligned} &p(x,y,t) = \text{constant, for } x,y \text{ satisfying,}\\ &x'(t) = -\beta y, \qquad y'(t) = -\alpha x, \text{ with,}\\ &x(0) = a_0, \qquad y(0) = b_0. \end{aligned}\right\} \tag{22}$$

At this point it is useful to examine the L equations corresponding to the expanded space SL equations. They are

$$x'(t) = -\beta y, \qquad y'(t) = -\alpha x$$

with ics

$$x(0) = ka_0, \qquad y(0) = kb_0.$$

All points on the k = 1 curves are transposed upward by a factor of k. Corresponding to $X_k(t)$ and $Y_k(t)$, x and y must be transformed, as follows, to $x_1 = x/k$, $y_1 = y/k$ to obtain the equations given above with x_1, y_1 replacing x,y and with ics, $x_1(0) = a_0$ and $y_1(0) = b_0$. So the proper equations to compare with equation (20) indeed are given by equations (22). Equations (22) are the standard L Square Law equations whose well-known solution shall be called $x_L(t)$, $y_L(t)$ with $x_L(0) = a_0$ and $y_L(0) = b_0$. This means that at every t in the interval $(0, t_f)$

$$p(x,y,t) = \delta(x-x_L)\,\delta(y-y_L). \tag{23}$$

Thus, it has been shown that $X_k(t)$ converges in distribution to $p(x,t) = \delta(x-x_L)$, that is, all the probability mass is concentrated at the point $x_L(t)$ for every t in $(0,t_f)$ and similarly

$$Y_k(t) \xrightarrow{D} \delta(y-y_L).$$

Now, when a rv converges in distribution to a <u>constant</u> that implies the stronger condition that it converges in <u>probability</u> to the same constant, see [11, p. 246]. Therefore, for $0 < t < t_f$,

$$\left.\begin{aligned} X_k(t) &\xrightarrow{P} x_L(t), \\ Y_k(t) &\xrightarrow{P} y_L(t) \quad . \end{aligned}\right\} \tag{24}$$

Furthermore, in the transformed space of the rectangle $\{[0,a_0]\times[0,b_0]\}$, a statement concerning how expected values $E[X_k(t)]$ and $E[Y_k(t)]$ behave can be made. It comes from a consideration of the convergence in probability equations (24) which, when written in terms of the definition of convergence in probability, become (for the A side only)

$$\lim_{k\to\infty} P[|X_k(t) - x_L(t)| < \varepsilon] = 1. \tag{25}$$

The interpretation of equation (25) is that in the limit as $k \to \infty$ the total mass of the sequence of random variables $\{X_k(t)\}$ becomes concentrated at point $x_L(t)$. Now, the $X_k(t)$ can have positive probability only at discrete lattice points 0, 1/k, 2/k,..., a_0 in the bounded interval $[0,a_0]$. Thus, as $k \to \infty$, any probability at these lattice points (unless $x_L(t)$ happens to be a lattice point for certain k values) approaches zero. Therefore, the first moment contribution about zero from any mass at lattice points approaches zero (except, of course, if $x_L(t)$ happens to be at a lattice point for certain ks). Clearly, then there is a very important conclusion in the <u>transformed</u> space that

$$\lim_{k\to\infty} E[X_k(t)] = x_L(t). \tag{26}$$

From equation (26) it is clear that $E[X_k(t)] = x_L(t) + \varepsilon(k)$, where $\lim_{k\to\infty} \varepsilon(k) = 0$. Thus, taking expected values in equation (15) yields

$$\frac{E[A_k(t)]}{k} = x_L(t) + \varepsilon(k)$$

or

$$\frac{E[A_k(t)]}{kx_L(t)} = 1 + \frac{\varepsilon(k)}{x_L(t)}$$

and

$$\left.\begin{aligned} &\lim_{k\to\infty} \frac{E[A_k(t)]}{k\ x_L(t)} = 1. \quad \text{Similarly for } Y_k(t), \\ &\lim_{k\to\infty} \frac{E[B_k(t)]}{k\ y_L(t)} = 1 . \end{aligned}\right\} \tag{27}$$

Equations (24), (26) and (27) are the principal focus of this section. (24) and (26) are a Law-of-Large-Numbers type result for the Lanchester Square Law. It is important to note that these two equations say that the transformed SL rvs converge in probability to the L Square Law which is the limiting mean. Contrary to many statements in the literature (both explicit and implicit) this does not say there is convergence of the means (i.e. mean SL may not tend to L) in the untransformed space. All that can be said about the untransformed SL sequence is given in equations (27). It has been noted previously that for $A_k(t)$, $B_k(t)$ that the corresponding L results are $kx_L(t)$ and $ky_L(t)$. Thus, equation (27) says that the ratio of expected values of SL to corresponding L equations go to a limiting value of 1. This does not necessarily mean that $[E[A_k(t)] - kx_L(t)]$ goes to zero in the limit. As a matter of fact, this difference may go to a constant (including zero) or infinity and the ratio still go to 1. The authors believe that this important distinction is pointed out here for the first time in Lanchester literature. Nothing that has been done to date says anything more about the limiting behavior of the untransformed difference.

Equation (20) has been derived in several other less rigorous, intuitive ways which seem to have concealed its true message as given above. These shall now be briefly examined.

B. Diffusion Approximations

Equation (20) can be considered a first order diffusion theory approximation to the SL Square Law process and it (or its implications) have been arrived at in several nonrigorous ways.

1. Taylor's Series Expansions

First is the Taylor's series approximation (see [7]). Again the SL Square Law is used to illustrate. The notion involved here is to replace the discrete function given by equation (17) (with k=1), by a continuous function that goes approximately through each of the discrete values of p(a,b,t) on the a,b axes at each time t. Thus, replacing discrete a,b by continuous x,y yields

$$\begin{aligned} \frac{\partial p(x,y,t)}{\partial t} &= \alpha x[p(x,y+1,t) - p(x,y,t)] \\ &\quad + \beta y[p(x+1,y,t) - p(x,y,t)] \quad . \end{aligned} \tag{28}$$

Now expand p(x,y+1,t) in a Taylor's series in powers of 1 around y and p(x+1,y,t) in powers of 1 around x and retain only the first order terms to get equation (20) immediately.

The usual explanation of when this (i.e. equation (20)) is a fair approximation to equation (17) is for αxt and βyt to be large. This is only speculation based on particular calculations. This really means for fixed α,β,t that x and y must $\rightarrow \infty$, which implies a transformation of the type previously discussed. Therefore, the usual assumption that this applies to the untransformed space is not correct. As was shown earlier the only correct conclusion on the untransformed space is given by equation (27).

2. Approximating Differences with Derivatives

Earlier Willard (1962 pp. 31-33) also arrived at equation (20) by a somewhat more intuitive approach where he divided the first term on the rhs of equation (17) by (b+1) - b and the second by (a+1) - a and called these Δb and Δa, respectively. He then replaced b + 1 by b + Δb and a + 1 by a + Δa and then assumed a and b were continuous and arrived at equation (20) by letting $\Delta a, \Delta b \rightarrow 0$. This, of course, is reasonable only for large a,b and again amounts to a mapping because Δa and Δb are always exactly 1 and can only be made a truly variable difference by a transformation. His solution is arrived at in a different manner but is (as it should be) equation (23). However, he incorrectly states that this implies that L is the limiting solution to untransformed SL. Again we reiterate the only conclusion on the basic SL equation is given by equation (27).

Koopman (1970, p. 87) and Taylor (1972, p. I-42) also used this technique to get equation (20) and solved it by the method of characteristics.

Helmbold (1966, pp. 632-635) also uses this technique for discrete state-space and discrete time parameter models of marksmen versus passive targets and many versus many battles to get L equations. This involves simultaneously passing to derivatives from differences on both state-space variables and time. The discrete time parameter in these models comes about because all contestants on a side fire in volleys at discrete time intervals.

3. Time-Independent State-Space Analyses

Williams (1963, p. 31, et. seq.) has written difference equations on the moments of the terminal survivor distribution as a function of the initial conditions for both the Square Law and the Linear Law. This is done by a random walk on the time-independent joint state-space. For example, he shows that for the Square Law

$$\mu_k(a_0,b_0) = \frac{\beta b_0}{\alpha a_0+\beta b_0}\,\mu_k(a_0-1,b_0) + \frac{\alpha a_0}{\alpha a_0+\beta b_0}\,\mu_k(a_0,b_0-1) \qquad (29)$$

where, μ_k is the k^{th} moment about the origin of the marginal distribution of the A side survivors (i.e., when A wins). He uses the Taylor's series expansion in powers of one (as explicated earlier in a different context) and again retains only first order terms to get the diffusion expression

$$\beta b_0 \frac{\partial \mu_k(a_0,b_0)}{\partial a_0} + \alpha a_0 \frac{\partial \mu_k(a_0,b_0)}{\partial b_0} = 0.$$

It is recognized, of course, that this really represents an equation for a Karr-type transformation and that a_0 and b_0 should be replaced by variables on the transformed space. He then solves this to show that μ_0 = Prob [A wins] = 1, for all real $\mu_1(a_0,b_0) = \sqrt{a_0^2 - (\beta/\alpha)\, b_0^2}$ and $\mu_2 = \mu_1^2$; therefore the variance is zero and furthermore all higher moments around the mean are zero. These equations are valid only for $a_0\sqrt{\alpha} > b_0\sqrt{\beta}$ of course. Similar expressions can be arrived at for the case where B wins. This merely shows that on the transformed space the terminal distribution is the terminal Lanchester point as one expects from the Karr-Etter development.

He shows the same thing for the Linear Law.

Covey (1969 p. 31) has shown that for the Linear Law, $\Delta m_A(t)/\Delta k = -\beta/(\alpha+\beta)$, where k is the number of the transition on the marginal time-independent state-space points. This is exactly true and shows that the marginal mean decreases linearly on equally spaced marginal points that lie exactly on the L Linear Law solution in state-space. However, at this point he makes the grossly incorrect inference that (for large numbers) the expression above may be approximated by $dm_A(t)/dt = -\beta/(\alpha+\beta)$ which is, of course, not true since it implies jumping from time-independent state-space to the time trace which is totally unjustified and, of course, does not give the correct result.

4. Miscellaneous Analyses

(1) Gye and Lewis (1974 pp. 6-7) give a curious twist to all this by starting with the usual spurious notion that equation (20) applies to the untransformed space and then applying the fundamental calculus identity, $dp/dt = \partial p/dt + (\partial p/\partial x)(dx/dt) + (\partial p/\partial y)(dy/dt)$ to arrive at the conclusion that the Square Law Lanchester equations are really the modal trace rather than the expected value trace in the untransformed space. It has been shown earlier that it is not the expected value trace and this certainly does not establish it as the modal trace as equation (20) only applies to the transformed space in the limit and no meaning can be attached to the expressions dx/dt and dy/dt in the untransformed space. In the transformed space it has been shown that all probability is located on the Lanchester trace and, therefore, all moments, etc. are located there and to say that the mode is located there adds nothing.

(2) Farrell (1976, pp. 4-13) has made an attempt at bounding the solutions to the mean value functions for the SL Square Law. In the first part of his paper he erroneously states that the L equations are lower bounds on the SL mean value functions ($m_A(t)$ and $m_B(t)$) for all a_0, b_0, $t < t_f$, α,β. This is patently not true from the discussion in Section II and from numerous counter examples. He then extends this to the Linear Law by some assumptions which are probably true but leave him with the same defect that vitiated his conclusions in the Square Law. Farrell's argument is based on the fact that equations (2) for the SL mean values would look exactly like the Lanchester equations if it were not for the two nonnegative terms on the right hand sides involving the absorption probabilities. For any positive t, these probabilities become positive and one might quickly conclude that the SL solutions are upper bounds to the L survivor functions. However, because of the interaction between the equations of each pair this conclusion is not true; and in fact, there are many counter examples.

In the next part he attempts to develop upper bounds on these same functions. Again there is a flaw which vitiates the results. He makes a transformation which does not require the initial force sizes to go to infinity at the same rate and thereby destroys the validity of the work. This can be demonstrated as follows. The Farrell transformation is given as

$$X(t) = A(t)/a_0 \text{ and } Y(t) = B(t)/b_0.$$

When this transformation is applied to equation (17) with $k = 1$, the result is

$$\begin{aligned}\frac{\partial p(x,y,t)}{\partial t} &= \alpha x a_0 [p(x,y + 1/b_0,t) - p(x,y,t)] \\ &\quad + \beta y b_0 [p(x + 1/a_0,y,t) - p(x,y,t)] \\ &= \alpha x \frac{a_0}{b_0} \frac{[p(x,y + 1/b_0,t) - p(x,y,t)]}{1/b_0} \\ &\quad + \beta y \frac{b_0}{a_0} \frac{[p(x + 1/a_0,y,t) - p(x,y,t)]}{1/a_0} .\end{aligned}$$

Now let a_0/b_0 = a constant say, c, and let a_0 and $b_0 \to \infty$ (using the same arguments about cells on the transformed state space as before) the result is

$$\frac{\partial p(x,y,t)}{\partial t} = \alpha x c \frac{\partial p(x,y,t)}{\partial y} + \beta y \frac{1}{c} \frac{\partial p(x,y,t)}{\partial x} . \tag{30}$$

Now, as long as c is held fixed this transformation will give the same results as before because the corresponding transformed L equations are

$$dy/dt = -\alpha c x, \qquad dx/dt = -(\beta/c)y$$

with ics

$$x(0) = y(0) = 1,$$

which indeed give the correct x_L, y_L coordinates for the solution to equation (30) which is again $p(x,y,t) = \delta(x - x_L)\,\delta(y - y_L)$. However, in all his conclusions, Farrell lets one side have a fixed initial condition and lets the other sides's ic $\to \infty$. This gives either $c = 0$ or $c = \infty$ and the whole analysis collapses.

In fact, if we let $a_0 \to \infty$ then $c \to \infty$ and the transition probabilities downward will tend to one and leftward to zero and in the limit all probability mass will be concentrated at $p(\infty,0,t)$, for all t for <u>both</u> spaces.

The basic point is that in any transformation of this type, if the initial numbers on one side go to infinity both must do so and furthermore both must do so at the same rate.

IV. FALLACIES IN THE APPROXIMATION OF GENERAL RENEWAL PROCESSES BY SL PROCESSES

In all that has been said up to this point it has been assumed in the SL model that all interfiring times X_A, X_B are negative exponentially distributed (ned) rvs. This, of course, greatly simplifies matters as the ned rv has "no memory".

Two attempts have been made to justify using the ned SL model as a good approximation to the general renewal model (i.e. with general ifts, X_A, X_B), under certain circumstances. These two arguments are now examined in some detail.

A. The Individual Firer Argument

First, there is the individual firer argument which is widely used in this country and is based (see Bonder and Farrell (1970), pp. 84-86) on Blackwell's Theorem in renewal theory. This argument goes as follows:

(1) Observe each independent firer on one side, all with identical independent ifts which are general rvs, X with mean ζ and with kill probability p. Embedded in each of these processes is a killing process with mean ζ/p, which is also iid and therefore renewal theory is applicable. Set $\zeta/p = \mu$, for simplicity, and call the embedded stochastic process $Z(t)$.

(2) Each firer has a sequence of kills which are counted and which is a rv in time, $N(t)$, the counting distribution of $Z(t)$.

(3) The uncountably infinite ensemble of all realizations of $N(t)$ has a mean value function of time, $n(t)$, whose slope $\to 1/\mu$ as $t \to \infty$.

(4) (3) implies that $Z(t) \to$ ned with mean μ, which is not true (as shall be shown).

(5) (4) implies that a superposition of all firers on one side tends to have ned interkilling times. If (4) is not true, this is, of course, not true.

Figure 9 below shows a few of the uncountably infinite set of possible realizations of $N(t)$. Of course, at every time t, the mean is the average over the entire ensemble and is denoted $E[N(t)] = \overline{n}(t)$.

In Figure 10 we graphically show some definitions from ordinary (non-terminating) renewal theory and below are some theorems from ordinary renewal theory (see reference [8]):

(1) $\lim\limits_{t\to\infty} \overline{n}(t)/t = 1/\mu$ (Elementary Renewal Theorem), this means the time average (slope of the chord) of the mean value function tends to $1/\mu$.

(2) $\lim\limits_{t\to\infty} N(t)/t = 1/\mu$ with probability one, this means that the time average (chord) of every <u>evolution</u> tends to $1/\mu$ (not shown in Figure 10, see Figure 9).

(3) $\lim\limits_{t\to\infty} d\overline{n}(t)/dt = 1/\mu$ (Blackwell's Theorem or the Key Renewal Theorem) means that the instantaneous rate tends to $1/\mu$.

It is (3) above which has been invoked to justify using p_A/μ_A for α and p_B/μ_B for β in SL and L approximations to GR.

The fallacies in this assumption are several;

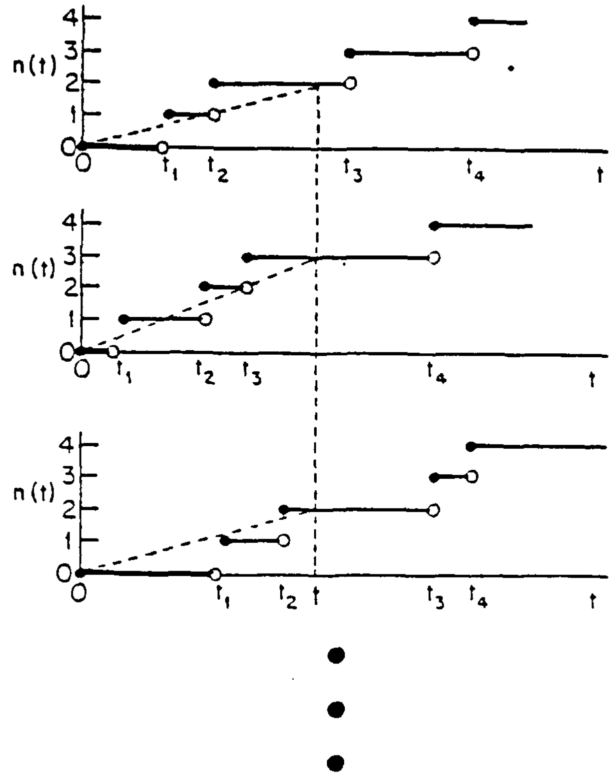


Figure 9. Realizations of the N(t) Process

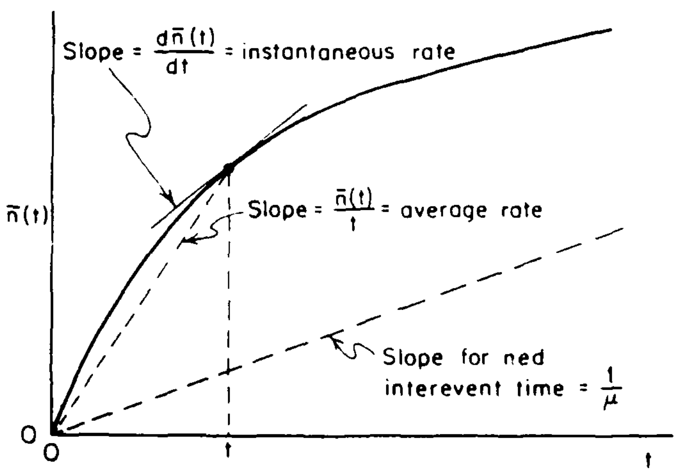


Figure 10. The Renewal Mean Value Function

(1) even if usable it would only apply after a "long" period of time has elapsed, and would certainly be erroneous in the early stages before "steady state" applies,

(2) $\bar{n}(t)$ is an ensemble average and therefore rates defined in properties (1) and (3) above are ensemble rates. They are a weighted average of all rates for certain special mutually exclusive and exhaustive subsets; weighted, as we shall see, by the probability that the backward recurrence time is some particular value. Thus, they cannot be used indiscriminately in probability calculations,

(3) none of these theorems apply to terminating processes (which are dealt with here), and thus none of this implies that the embedded killing process is tending to ned with mean μ. In fact, for the classical terminating process it is easy to show that $\lim_{t\to\infty} d\bar{n}/dt = 0$.

To further illuminate point (2) above, examine what the correct instantaneous rate is. The first thing to note is that except for ned interevent (interkilling) times it is not sufficient to simply specify $n(t)$ for the state. To completely and unambigously define the state, $n(t)$ and the backward recurrence time (y) must be specified (see Figure 11). This is because, in general, the system has Markovian "memory" and it does "remember" the time of the last event.

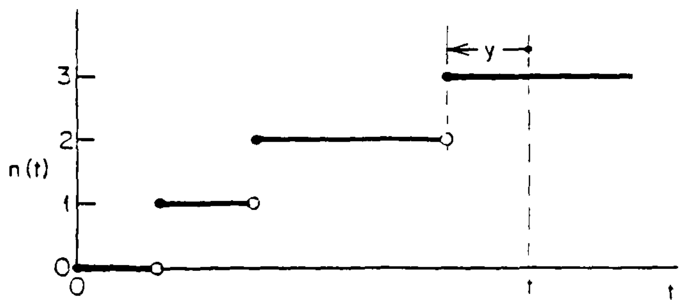


Figure 11. The Backward Recurrence Time

The rv Y with value y at t, as shown, is the necessary additional information to completely specify the state. A function of y, r(y) is the ensemble rate for the subset of all realizations with Y = y and can be used for probability calculations, i.e. $r(y)\Delta = P\{\text{event (kill) in } (t,t+\Delta)\}$ where $r(y) \triangleq f_Z(y)/F_Z^c(y)$ which is the instantaneous rate given Y = y and is the weighting factor referred to in (2) above.. Note that if Z is ned then $r(y) = 1/\mu$ and does not depend on y.

B. The Superposition Argument

Now examine the second argument which is called the superposition argument and which is widely used in the Russian literature, Ventsel (1964). This argument boils downs to the Palm-Khintchine theorem, see reference [9], which essentially states that if all a_0 iid interkilling processes, Z_A, on say the A side, are superposed to form a new interkilling process, $W_k(t)$, then $W_k(t) \to$ ned with mean $1/c$, as $a_0 \to \infty$ if

$$(1) \quad \sum_{i=1}^{a_0} 1/\mu_A = c, \text{ and}$$

(2) each $\mu_A \to \infty$,

where the μ_As are the means of the iid Z_As.

In words this simply means that for either side (say the A side) the superposed embedded killing process tends to ned as the number of simultaneous firers tend to infinity if each firer's interkill time mean tends to infinity. No one has investigated how large a_0 must be and how large μ_A must be for this to be practically useful.

It should also be noted that this theorem applies to non-terminating processes (no theorem like this for terminating processes is known to the authors). Indeed, obviously, as time progresses in a terminating situation the requirement for large numbers will sooner or later be badly violated.

It should be noted here that the only free world example of this misapplication of the Palm-Khintchine Theorem that we have found is in Cho (1984) where he incorrectly assumes it is applicable in a multiple marksmen versus multiple passive targets situation.

C. Firing and Killing Rates

Next, an interesting controversy which continues to crop up from time to time (see reference [4]) is considered. It is the question of the appropriate measure for the individual firing rate (this is equivalent to the question of the measure on the individual kill rate as it is easy to prove that in the GR model, say for the A side, the individual kill rate is $p_A r_A$, if $r_A = 1/\mu_A$ is defined as the individual firing rate). The question which has been raised is, what is correct, $r_A = 1/E[X_A]$ or $r_A = E[1/X_A]$? Bonder [5] originally proposed that $E[1/X_A]$ was the proper rate but Barfoot [2] made an intuitive argument that $1/E[X_A]$ is the correct one and implicitly (though not explicitly) Bonder [6] finally agreed.

The fact is that, in general, as has been shown above, neither is correct and there is no such general fixed rate. However, if one must use such an approximation there is no question that $1/E[X_A]$ is the better one. Renewal Theory (see above) shows that for the SL process it is exact and is therefore a counter-example to the original Bonder thesis; for the GR process it is asymptotically correct. From an intuitive viewpoint what the controversy boils down to is the following; suppose one collects n independent interevent times x_i, $i = 1,2,\ldots n$. Should one consider the sample event rate to be $n/\sum_{i=1}^{n} x_i = 1/\left(\sum_{i=1}^{n} x_i/n\right) = 1/\widehat{E[X_A]}$ or should one consider each $1/x_i$ to be a sample of the event rate leading to the sample event rate $\left(\sum_{i=1}^{n} 1/x_i\right)/n = \widehat{E[1/X_A]}$? Barfoot pointed out that the first is an event average and the second is a time average and the first is what we seek. We note, in passing, that for a positive rv, X, there is a theorem that $E[1/X] > 1/E[X]$ (see reference [11], p. 166). It is time to bury this controversy.

D. Further Comments

(1) It is important to note that there are theorems which state that, in general, even for non-terminating processes, the interevent times for superposed renewal processes are

(a) not identically distributed and,

(b) are not independent,

both of which are necessary for renewal theory to apply to any process. The only exception is for ned processes.

(2) It should not be implied that if, in either the L or SL formulation, the kill rates α, β are made functions of time that the resulting situation may drive these formulations closer to the GR model. What happens if this is done is that in the SL formulation each individual firer's process becomes a non-homogeneous Poisson process which is indeed Markovian but renewal theory does not apply as the interkill rv depends on time (i.e. they are not identically distributed) and successive interkill times are dependent. Thus the renewal character of the individual processes of the model are destroyed and although it is a model of some process it is not necessarily driving SL closer to GR.

The only possible function which might have some merit in this regard is using $d\bar{n}_A(t)/dt$ for α and $d\bar{n}_B(t)/dt$ for β, for very large a_0, b_0. This has not been investigated. The notion here is that for sufficiently large a_0, b_0, in the superposed process there may be enough backward recurrence times of various sizes and in appropriate proportions to imitate the ensemble mentioned earlier at every t. Again though, the imitation will surely become poor as time increases.

(3) A typical result from Gafarian and Ancker (1984) is given in Figure 12 which shows a wide discrepancy in comparable L, SL, and GR model mean value functions. This is a two-on-one situation with Erlang (2) ifts on the A side and ned on the B (one) side.

V. THE IMPORTANCE OF PROCESS VARIANCE

Up until now, the focus has been upon the notion that certain combat processes are adequately described by their mean value functions and in particular, by a deterministic Lanchester approximation to this function.

A. The Initial SL Variance

Implicit in the idea mentioned above is the assumption that (at least for large numbers on each side) the variance of the process is unimportant. This assumption is largely based on Brooks (1965). Brooks has looked at processes such as ours in state-space and examined them at the lattice points only and in particular at the lattice points where successively 1 event only has occurred, 2 events only, 3 events only, etc. up to the point where the number of events (k) is equal to or less than the min (a_0,b_0) (see Figure 4.) This ensures that not enough events (events being kills on either side) have occurred for an absorption to have occurred. This essentially means very early in the combat. He then defines a concept called "stochastic determinism" as the property of a process at the k^{th} event, that the quantity, σ_k/a_0, is small for all $k \leq \min(a_0,b_0)$ and where σ_k is the standard deviation of the losses on the A side (this is also the standard deviation of the survivors).

Next he shows for the SL Linear Law (in general, even though he claims it for a succession of one-on-one duels only) that $\sigma_k/a_0 < 1/2\sqrt{a_0}$, $k \leq$ min

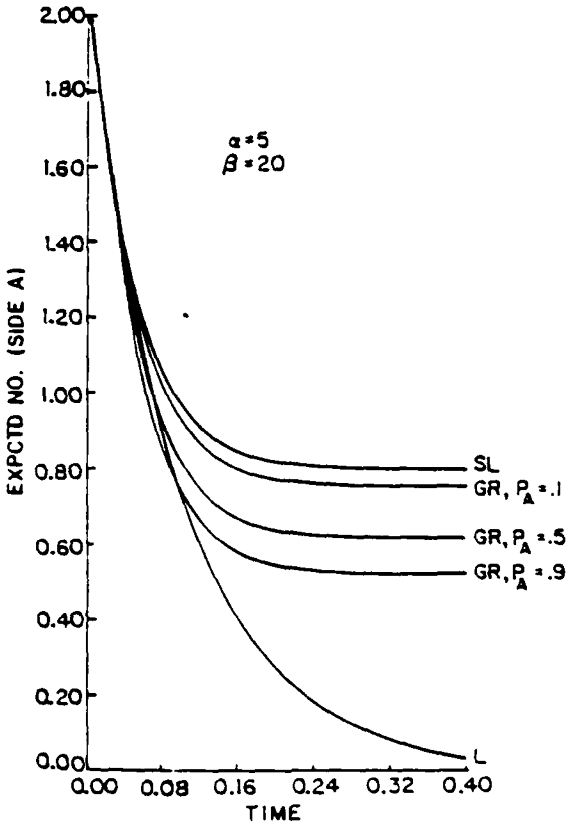


Figure 12. An Example of Comparable L, SL, and GR Model Mean Value Functions.

(a_0, b_0), which is sufficent for his definition of stochastic determinism. However, examination of the coefficient of variation of the losses for this case yields $\sqrt{\alpha/k\beta}$, or a minimum value of $\sqrt{1/\min(a_0,b_0)}\ \sqrt{\alpha/\beta}$. Clearly, the term $\sqrt{\alpha/\beta}$ can be very large indeed and one would surely not be justified in calling this quantity small for all possible parameter values. Or, just look at the value of σ_k, which is $\sqrt{k\alpha\beta}/(\alpha+\beta)$ and which has a maximum of $(1/2)\sqrt{k}$. Again for large values of a_0 and b_0, this can attain a large value since k can equal $\min(a_0,b_0)$.

For the Square Law Brook's results are similar (he only considers the case where $\alpha = \beta$).

Clark (1969, pp. 132-133) using a technique suggested by Snow (1949) has given the variance for the SL Square Law. The technique solves the variance equations again using the assumption that the absorption probabilities are zero. This is essentially the same idea as Brooks above, but gives the more informative time trace (good only for times near zero). The equations clearly indicate that although the percentage losses, for large a_0 and b_0 may be small, the absolute values may be large. This supports the analysis above.

Willis (1982, p. 6) has arrived at exactly the same ordinary differential equations on the moments as Clark (1969) above. His technique is to use the differential-difference equation (17) for interior points to obtain a partial differential equation on the joint moment generating function of the process. The moment generating function is then expanded in a Taylor's series in powers of the two transform variables and coefficients of like powers on each side of the equation are equated to generate the ordinary differential equations. Since boundary equations are ignored this again implies absorption probabilities are zero and is good only near time zero. For some models this technique may be easier than Clark's and Snow's.

Weale (1972 pp. 11-12) reproduces Clark's results (apparently unknowingly). However, he does give an approximate expression based on a Normal approximation to the joint pmf of A(t) and B(t) for the time (measured from t = 0) during which Clark's expressions are good to any desired degree of approximation. (The relevant equations are (42), (43) p. 20, (38) and (39) p. 18 and, (19) p. 8 in Weale.) The general idea is illustrated in Fig 13 below. Although Weale does this for a breakpoint analysis, the annihilation situation is simpler and illustrates the point. If $1 - \exp(-c^2/2)$ is the joint survival probability (for a_0, $b_0 \to \infty$) contained inside the contour c at some particular time for some particular probability then certainly some of the remaining probability is in the form of absorption probability on the a and b axes. Roughly if a total absorption probability is selected that is not so "large" as to distort the results then, one minus this probability is the probability desired inside the contour. Then, by a trial and error procedure the time at which the desired contour is tangent to

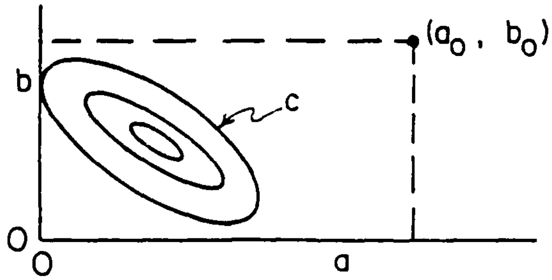


Figure 13. Survival Probability Contours

one or both of the axes is determined. In any event the time interval thus calculated is conservative.

Perla and Lehoczky (1977 pp. 5-12) derive a diffusion model for the SL Square Law. In this approximation the mean value is assumed to be the $x_L(t)$, $y_L(t)$ time-trace and the pdfs at every time, t, are continous and assumed Normal (invoking the Central Limit Theorem). Then the procedure derives the second moments. The variances are exactly as given by Clark (1969) above. There is also a covariance determined which Clark could have, but, did not, give. The Perla and Lehoczky model is only good until the absorption probabilities are significant, and is essentially no different than Clark (1969). Again they show that early variances may be large.

In general, the use of a continuous state-space implies a limiting mapping (good only on a tranformed space) as explained earlier. But, for approximations, these expressions with very large initial numbers are used in the untransformed state-space. This is justified by the following reasoning (see Figure 14 below).

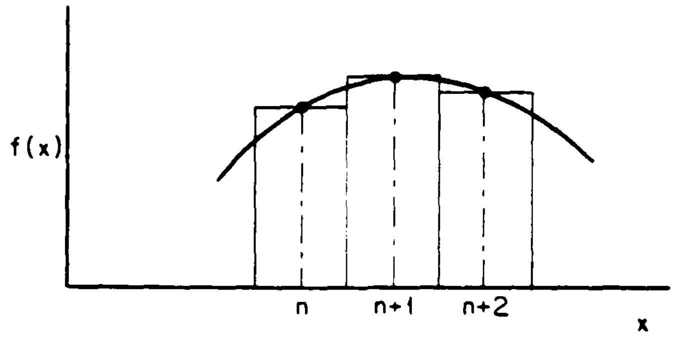


Figure 14. The Diffusion Approximation

The figure only shows a univariate situation but the reasoning carries over to the bivariate situation. The probability mass (such as at points n, n+1, n+2) is converted to a rectangular probability density around each mass point, as shown, with the same height, since the intervals are exactly one unit apart. The histogram thus created is approximated by a curve which nearly goes through the mass points as shown (it only goes through them exactly in the limiting transformed space). The mean value function is approximately the L equation with possibly large variance for very large initial condition and early in the process.

The conclusion is that "stochastic determinism" as defined by Brooks does not ensure that there are not large absolute variations, even in the early stages of SL models with large initial numbers on each side.

B. The Terminal SL Variance

It can be deduced from Weiss (1963) that for the SL Linear Law the terminal distribution is asymptotically normal. Also Gye and Lewis (1976) show asymptotic normality for the terminal distribution of the SL Square Law.

Gye and Lewis (1974, p. 19) have shown that for the Square Law the terminal distribution of A side survivors for large initial numbers is approximately Normal with a standard deviation of about $.35\sqrt{a_0}$ if the A side has overwhelming superiority and about $.37\ a_0^{3/4}$ for $a_0 = b_0$. In both cases $\alpha = \beta = 1$.

Watson (1976) uses Martingale Theory to arrive at similar terminal results. This involves a transformation of A(t) and B(t) to get a Martingale rv whose terminal properties are easily arrived at. However, the inversion to get terminal properties for A(t) and B(t) is far from simple and it is not obvious that there is any computational gain.

It is clear from many examples that terminal variances for initial conditions of any size are always substantial.

C. The Transient SL Variance

Taylor (1972), (I-45,46) develops a second order diffusion approximation to the SL Square Law by standard diffusion techniques. This partial differential equation has unknown coefficients; however, the following technique identifies the coefficients.

The procedure is the same as given previously starting with equation (28) and expanding the p(x,y,t)s in powers of one around x and y, except that this time one keeps second order terms. It is simple to show that one immediately gets

$$\frac{\partial p(x,y,t)}{\partial t} = \alpha x \frac{\partial p(x,y,t)}{\partial y} + y \frac{\partial p(x,y,t)}{\partial x}$$

$$+ \frac{1}{2} \left(\alpha x \frac{\partial p^2(x,y,t)}{\partial y^2} + \beta y \frac{\partial^2 p(x,y,t)}{\partial x^2} \right),$$

with initial condition

$$p(x,y,0) = (x-x_0) \quad (y-y_0).$$

This equation has not been solved at this time, and so adds little to our knowledge. One is tempted to hope that its solution might be a Normal pdf with mean $x_L(t)$, $y_L(t)$ and variances and covariance as given by Clark (1969) and Perla and Lehockzy (1977), however, this does not appear to be the case. In fact, the process of a Taylor's expansion in powers of one seems to give correct results (on a transformed space) for first order (mean value) results but appears to break down for second order results. The second order term $\partial^2/\partial x \partial y$ is always missing and seems necessary. This may be due to the fact that the term in one squared is not small compared to one to the first power.

Farrell (1976, pp. 25, 26) gives a much improved method of estimating Square Law variances by approximating absorption probabilities and thus allows the analysis to go beyond the initial stages. He gives an example where the variances at any time, t, are substantial.

Clark (1969 pp. 125, 126) has shown by examples that Square Law variances start at zero at time zero and tend to an asymptote at time infinity. In between they are either montonically increasing or, increasing then decreasing. Their values are substantial.

D. The GR Variance

Finally, we mention that the only known GR solution for more than one-on-one (for one-on-one, see reference [1]) is a stochastic duel with two versus one (see Gafarian and Ancker (1984)). A typical result comparing GR with SL is given in Figure 15. Clearly variation is important, and GR variance may differ considerably from SL variance.

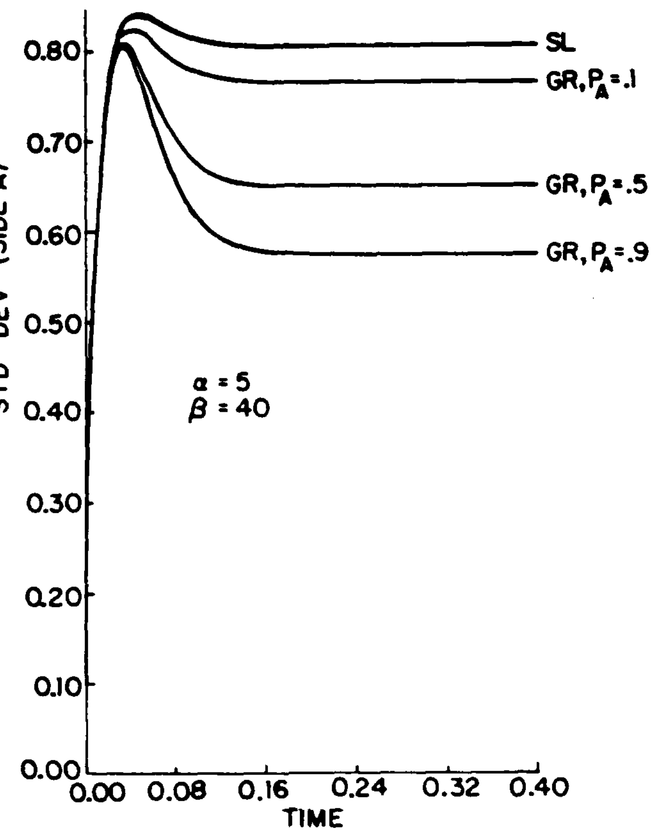


Figure 15. Two-on-One Stochastic Duel with Erlang (2) on the A (Two) Side and Ned on the B Side

This section is concluded with the observation that it can surely be said that variance is too important to be ignored in any realistic interpretation of combat models. This is a strong argument for rejecting the notion that some deterministic approximations of the mean value of the stochastic process is sufficient in combat analyses.

VI. THE ERRORS IN OTHER MEASURES OF EFFECTIVENESS

Up to this point the principal concern has been with the mean value trace of the survivors as a measure of combat progress. However, there are three other measures of combat outcome which are at least crudely predicted by the L equations and which are now examined, especially in regard to their relationship to the corresponding SL measures. No attempt to compare with the equivalent GR measures will be made here. These measures are; expected number of survivors, expected time-duration of the battle and the probability of winning. These three predictors, with the previously discussed mean value trace are the only possible direct measures that can be obtained from the L model.

A. The Expected number of survivors

1. The Linear Law

Weiss (1963, p. 598) has shown that the conditional SL-L mean value difference at infinity (i.e. given a win by A) is

$$\Delta_{A|A}(\infty) = \frac{m_A(\infty)}{P(A)} - \frac{x(\infty)}{1} = a_0 \binom{a_0 + b_0 - 1}{a_0} \left(\frac{\beta}{\alpha + \beta}\right)^{a_0} \left(\frac{\alpha}{\alpha + \beta}\right)^{b_0 - 1} \times \frac{1}{P(A)},$$

where $a_0 > (\beta/\alpha)b_0$. Thus L underestimates SL as this is always positive. The corresponding B side difference is meaningless since marginal $y(\infty) = 0$ and the P(B) for L is also zero and their ratio is indeterminate. Of course, by symmetry the same can easily be done for wins by B when $a_0 < (\beta/\alpha)b_0$.

However, the marginal difference, which is the appropriate one to compare, is not so clear cut. For example,

$$\Delta_A(\infty) = m_A(\infty) - x(\infty) = a_0 \binom{a_0 + b_0 - 1}{a_0} \left(\frac{\beta}{\alpha+\beta}\right)^{a_0} \left(\frac{\alpha}{\alpha+\beta}\right)^{b_0 - 1} - P(B)\left(a_0 - \frac{\beta}{\alpha} b_0\right),$$

again with $a_0 > (\beta/\alpha)b_0$. This will be very difficult to explore except by particular examples which is shown in Part Two.

2. The Square and Mixed Laws

No closed form expression for the SL mean value functions at $t = \infty$ exist although useful forms for the marginal distributions of survivors do.

In general, L always predicts the loser with zero expected survivors and near parity this can be very misleading. The discrepancy on the winner's side can best be investigated by examples as seen in Part Two.

It has been clearly shown that the L predictor can be extremely misleading.

B. The Expected Time-Duration of the Battle

Very little has been done theoretically on this measure. However, in Square Law parity and always in the Linear Law battle $t_f \to \infty$ for L. This is clearly a useless result as all SL battles have a finite expected time-duration because,

$$f_{T_D}(t) = \sum_{a=0}^{a_0} p'(a,0,t) + \sum_{b=0}^{b_0} p'(0,b,t)$$

where f_{T_D} = a proper pdf of T_D (the rv, time-duration of the combat). The rhs is, in general, a weighted finite sum of exponentials and will therefore have a finite mean.

Bowen (1965) has shown that for any SL model, the mean time to the <u>first</u> kill is less than the corresponding L time. While not conclusive concerning the overall battle, this is still an interesting and suggestive result.

Some particular examples have been worked out or obtained by simulation. These results are given in Part Two and in general again the L prediction can be very misleading.

C. The Probability of Winning

1. The Linear Law

Brown (1951 and 1955) laid the ground work for the following expression which was later almost simultaneously put in finished form by Brown (1963), Williams (1963), and Weiss (1963) (most explicitly). For SL,

$$P(A) = I_{\alpha/(\alpha+\beta)}(b_0, a_0), \qquad P(B) = I_{\beta/(\alpha+\beta)}(a_0, b_0),$$

where I is the Incomplete Beta Function Ratio. For L

$$P(A) = 1, \qquad P(B) = 0, \qquad a_0 > (\beta/\alpha)b_0,$$

$$P(A) = 0, \qquad P(B) = 1, \qquad a_0 < (\beta/\alpha)b_0,$$

$$P(A) = P(B) = 0, \qquad P(D) = 1, \qquad a_0 = (\beta/\alpha)b_0,$$ where P(D) is a disastrous draw. The last expression seems an appropriate interpretation since both sides go to zero survivors at $t = \infty$, and neither can be said to have won.

From P(A) for SL we observe that $0 \leq I_{\alpha/(\alpha+\beta)}(b_0, a_0) \leq 1$ and that for fixed b_0 and a_0 it is monotonically increasing as $\alpha/(\alpha+\beta)$ increases or for fixed α, β, b_0 it is also monotonically increasing with a_0. Figure 16 is a typical comparison with L.

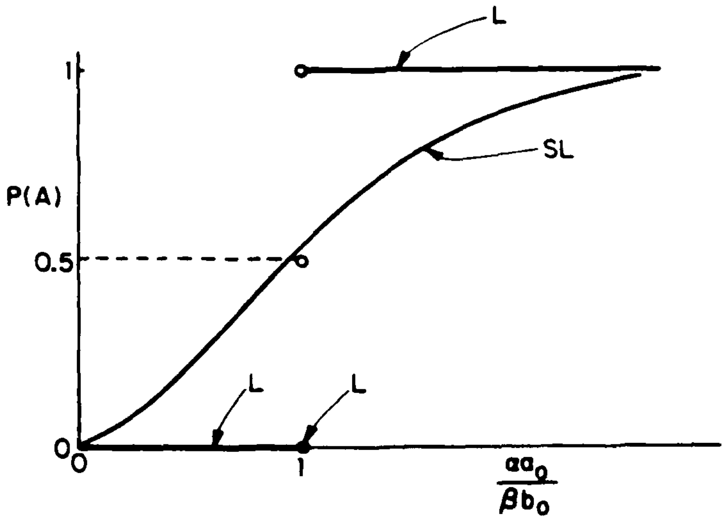


Figure 16. Probability of Winning for L and SL

Note that this general curve has not been shown going through $P(A) = 0.5$ at $\alpha a_0/\beta b_0 = 1$ because this will only occur for strict SL parity (i.e. $a_0 = b_0$, $\alpha = \beta$), otherwise it may cross either above or below depending on the exact values of the parameters. Such curves are obtained by fixing a_0, b_0 and either α or β and varying the other. It is striking that L can imply P(A) is zero when it is > 1/2 or P(A) is 1 when it is less that 1/2. The P(B) situation is, of course symmetrical.

2. The Square Law

The SL probability of winning is also given in Brown (1951, 1955 and, 1963) and Isbell and Marlow (1956) as a complicated sum. Although, in fact, it does have quite similar properties to the Linear Law (as shown by many examples in Part Two) it is very difficult to observe its properties theoretically.

Brown (1951, 1955 and 1963) has derived useful approximations for both the SL Linear and the Square Laws which are Normal probability integrals, derived asymptotically but which are remarkably good for small numbers. Kisi (1966) has derived the same expression for the Square Law only by a simple transformation of the basic differential-difference equation which captures second order differences (i.e., differences of differences) and then replaces differences by derivatives.

It can be said generally, that near parity the L predictor for winning can be extremely misleading. In Part Two this is illustrated with particular examples.

VII. CONCLUSIONS

In the following we summarize the main points in Part One:

(1) All L-SL mean value equivalent pairs differ (possibly considerably) for all times except at crossing points.

(2) At least for the Square Law, the L trajectories are neither a universal upper or lower bound on the SL mean value trajectories.

(3) Even near time zero, the L and SL mean value trajectories may differ considerably (they do not differ materially for the Square Law and the sequence of one-on-one duels version of the Linear Law.)

(4) For the Linear Law, the Square Law, the Mixed Law and the Square Law with continuous reinforcements there is a Law of Large numbers on suitably transformed spaces. However on untransformed spaces one can only say $\lim_{k\to\infty} E[A_k(t)]/k\, x_L(t) = 1$, $\lim_{k\to\infty} E[B_k(t)]/k\, y_L(t) = 1$. This does not necessarily mean that as the initial force sizes tend to infinity the differences between L and SL mean value trajectories tend to zero. They may even tend to a constant or infinity.

(5) Blackwell's Theorem does not imply that individual combatants (and thus their superposition) with general ifts tend to have ned ifts (even after a long time). This is even more strongly the case for terminating processes.

(6) The Palm-Khintchine Theorem does not imply that superposing a large number of combatants with general ifts will yield a process with ned ifts. This can only be approximately correct for large numbers and for very large ift means. Again the theorem is only valid for non-terminating processes.

(7) Nonhomogeneous Poisson processes do not, in general, approximate general renewal processes.

(8) The SL process variances are generally quite significant and can be important for large force sizes, even near time zero. In addition, GR process variances are significantly different than SL process variances.

(9) The other L measures, (a) expected number of survivors, (b) expected time duration of the battle and (c) probability of winning are even less reliable predictors than the mean value trace.

(10) Finally, we emphasize that the basic assumptions of the SL (and GR for that matter) models simply can not hold for large numbers of combatants. Terrain compartmentalization, weapon ranges, terrain obstacles, weather and many other factors (including tactical ones) cause large scale battles to be a set of sequential and/or parallel small scale engagements. The effect of this point is illustrated in Figure 17, where one large battle with 96 on each side is compared to 16 simultaneous battles of 6 on each side. The Lanchester solution is identical for both cases but the SL solution is quite different

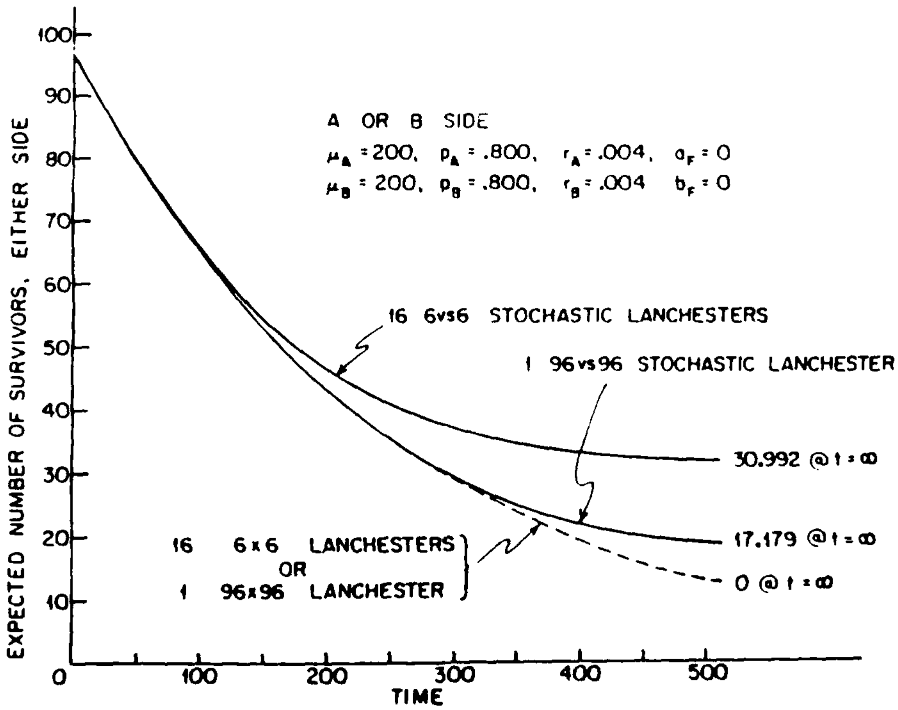


Figure 17. A Comparison of One Large Battle with Several Simultaneous Smaller Battles

for the two cases with the terminal number of survivors in the simultaneous small battles being 80.4% greater than the one large battle. Further results (Ancker and Gafarian (*)) are contained in Section 6, Part Two. Thus, the development of small scale GR models and their integration into large models is intrinsically necessary. The effect of this fractionation on overall measures of outcomes is unknown. For a discussion of current thinking on this matter in England and the United States see reference [3].

VIII. RECOMMENDATIONS

Both analytical (mathematical) and simulation research is needed and is recommended on the following topics on GR models:

(1) Solutions for moderate size a_0 and b_0.

(2) Good approximations for moderate size a_0 and b_0.

(3) Superposition of terminating renewal processes (this would probably be best started on non-terminating processes and proceed towards terminating processes later).

(4) The possibility of using $d\bar{n}(t)/dt$ as the instantaneous rate for moderate numbers of superposed iid GR processes.

(5) Numerical techniques to solve the complicated analytical models.

(6) In simulations, variance reduction techniques on non-classical terminating processes.

(7) Determination of error bounds on approximations.

(8) Integration of small (or moderate) size battle models into large models of combat.

LIST OF REFERENCES

[1] Ancker, C.J., Jr., One-on-One Stochastic Duels, Research Monograph published by the Military Applications Section of the Operations Research Society of America, Jan 1982, 381 pp.

[2] Barfoot, C.B., "The Lanchester Attrition-Rate Coefficient: Some Comments on Seth Bonder's paper and a Suggested Alternate Method", Operations Research, Vol. 17, No. 5, Sept 1969, pp. 888-894.

[3] Bathe, M.R., "Modeling Combat as a Series of Mini Battles", TRASANA-LR-14-84, US Army TRADOC Systems Analysis Activity, White Sands Missile Range, NM 88002, 27 July 1984, 16 pp.

[4] Bathe, M.R., Haysman, P.J. and Smith, W.E., "Determination of the Expected Time-to-Kill for a Direct-Fire Engagement", Technical Note OR/TN/14, Royal Military College of Science, Shrivenham, UK, Sept 1983, 12 pp.

[5] Bonder, S., "The Lanchester Attrition-Rate Coefficeint", Operations Research, Vol. 15, No. 2, Mar-April 1967, pp. 221-232.

[6] Bonder, S., "The Mean Lanchester Attrition Rate", Operations Research, Vol. 18, No. 1, Jan-Feb 1970, pp. 179-181.

[7] Daniels, H.E., "Approximate Solutions of Green's Type for Univariate Stochastic Processes", Journal of the Royal Statistical Society, Series B, Vol. 22., 1960, pp. 376-401.

[8] Karlin, S. and Taylor, H.M., A First Course in Stochastic Processes, 2nd Edition, Academic Press, 1975.

[9] Khintchine, A.Y., Mathematical Methods in the Theory of Queuing. Hafner Publishing Co., N.Y., 1960 (a translation from the Russian by D.M. Andrews and M. H. Quenouille).

[10] Lanchester, F.W., Aircraft in Warfare: The Dawn of the Fourth Arm. Constable, London, 1916.

[11] Rohatgi, V.K., An Introduction to Probability Theory and Mathematical Statistics, John Wiley and Sons, 1976.

ANNOTATED BIBLIOGRAPHY

ABREVIATIONS:

DOAE	Defence Operational Analysis Establishment, West Byfleet, UK.
DTIC	Defense Technical Information Center, Alexandria, VA 22314.
IDA	Institute for Defense Analyses, Arlington, VA 22202.
MIT	Massachussetts Institute of Technology, Cambridge, MA 02139.
MOD	Ministry of Defence, West Byfleet, UK.
NATOCON	Paper presented at the Nato conference on "Recent Developments in Lanchester Theory", Munich, West Germany, July 3-7, 1967.
NPGS	Naval Postgraduate School, Monterey, CA 93940.
NRLQ	Naval Research Logistics Quarterly.
OR	Operations Research, Journal of the Operations Research Society of America.
OSU	Ohio State University, Columbus, OH 43212.
RMCS	Royal Military College of Science, Shrivenham, UK.
UMic	University Microfilms International, Ann Arbor, MI 48106.
UK	United Kingdom (England).

This bibliography annotates the cited papers only for those portions of the papers which contribute theoretical material or particular examples relevant to the purposes of this study. This by no means exhausts the total contributions of these works.

The annotations are coded as follows:

AT = Attrition study, Theoretical
AE = Attrition study, Examples for particular values of parameters.

I = Difference between SL mean value functions and L functions.
II = Expected number of terminal survivors.
III = Expected time-duration of the combat.
IV = Probability of winning.
V = Variance of the process.
VI = Miscellaneous results.

ANCKER, C.J., JR. and GAFARIAN, A.V. (*)

New results presented for the first time in the paper.

AT Compares L and SL battles of moderate size with several simulataneous L and SL battles with the same rates on the opposing sides and whose total initial numbers on each side equal the larger battle. Mean value functions and standard deviations are shown.

AE I,II, and VI pp. 183-192.

BONDER, S. and FARRELL, R. (1970)

"Development of Analytical Models of Battalion Task Force Activities", SRL 1957 FR 70-1 (U), Systems Research Laboratory, Department of Industrial and Systems Engineering, University of Michigan, Ann Arbor, MI 48106, Sept 1970, 640 pp. DTIC# AD 714677.

AT Invokes Blackwell's Theorem to justify using SL equations for GR model (pp. 84-86). This is an inappropriate usage.

BOWEN, K.C. (1965)

"A Comparison of the Duration of a Deterministic Battle and the Mean Duration of an 'Equivalent' Stochastic Battle", Research Working Paper L4, DOAE, MOD, UK, June 1965, 6 pp.

AT Shows that for any general SL model the mean time to the first kill is less than the corresponding L time.

BROOKS, F.C. (1965)

"The Stochastic Properties of Large Battle Models", OR, Vol. 13, No. 1, Jan-Feb 1965, pp. 1-17.

AT Defines "Stochastic Determinism" and shows that SL Square Law and Linear Law comply.

BROWN, R.H. (1951)

"The Solution of a Certain Difference Equation with Applications to Probability", Ph.D. Thesis in Pure Science, Columbia University, New York, NY, 1951, 40 pp. UMic #33251.

AT Derives a Normal integral approximation to the probability that the A (or B) side wins in an SL Square Law battle with large initial numbers.

BROWN, R.H. (1955)

"A Stochastic Analysis of Lanchester's Theory of Combat (U)", Technical Memorandum, ORO T-323, Operations Research Office, Johns Hopkins University, Chevy Chase, MD, Dec 1955, 30pp. DTIC# AD 82944.

AT Derives a Normal integral approximation to the probability that the A (or B) side wins in an SL Square Law battle with large initial numbers.

AE IV p. 29.

BROWN, R.H. (1963)

"Theory of Combat: The Probability of Winning", OR, Vol. 11, No. 3, May-June, 1963. pp. 418-425.

AT Derives Normal integral approximations to the probability that the A (or B) side wins in both SL Square Law and Linear Law battles.

CHO, J.K. (1984)

"Combat Attrition Analysis Using Renewal Process", MS Thesis in Operations Research, NPGS, March 1984, 55 pp. DTIC# AD-A143139.

AT Incorrectly applies the Palm-Khintchine Theorem to multiple marksmen firing at multiple passive targets.

AE I p. 48.

CLARK, G.M. (1969)

"The Combat Analysis Model", Ph.D. Thesis in Industrial Engineering, OSU, 1969, 286 pp. UMic #69-15905.

AT Shows that difference terms between all possible L function and equivalent SL mean value functions exist (pp. 78-81). Gives explicit expressions for the Linear (pp. 81-83) and Square Laws (p. 80) and a special acquistion model (p. 151). Gives variances for the SL Square Law (pp. 132-133) before absorption is significant.

AE I and II pp. 117, 118, 119, 121, 122, 153. V pp. 125-128, 134, 154.

CLARK, G.M. (1982)

"Low-Resolution Small Unit Model (LORSUM)", Report RF783217 FR 82-1 (U), Research Foundation, OSU, May 1982, 502 pp.

AE I and II pp. 157, 158, 160-169.

COVEY, R.W. (1969)

"Dropouts in Combat: A Stochastic Model", MS Thesis in Operations Research, NPGS, October 1969, 39 pp. DTIC# AD 704482.

AT Shows Linear Law SL mean coincides exactly with L at transition points in time-independent state space (p. 31).

CRAIG, J.D. (1975)

"The Effect of Uncertainty on Lanchester-Type Equations of Combat", MS Thesis in Operations Research, NPGS, September 1975, 174 pp. DTIC# AD A017550.

AT Integrals for the difference between the SL mean value functions and the L equations for the Square Law with breakpoints (pp. 65, 68) and with annihilation (p. 68).

AE I pp. 71-81, 83, 85-92, 94, 96, 98-126. IV pp. 44, 46-50. V pp. 29-33, 57-64.

DALY, F. (1984)

"The Multi-threat 'Square' Law Battle", Mathematical Scientist, Vol. 9, 1984, pp. 1-14.

AE IV p. 12.

DATHE, H.M. (1967)

"Some Extensions of Stochastic Combat Analysis", ORG/MS-19, Operations Research Gruppe der IABG, Munchen, West Germany, April 28, 1967, 26 pp. NATOCON.

AE IV pp. 18-23.

ETTER, D.O. (1971)

"Deterministic Combat Attrition Models for Spatially Distributed Forces", Paper P-577, Systems Evaluation Division, IDA, May 1971, 95 pp. IDA Log No. HO 69-10951.

AT Rigorous proof that transformed versions of both Linear Law and Square Law SL mean value functions converge in probability to transformed versions of L equations.

FARRELL, R.L. (1976)

"Measuring the Errors in Approximating Stochastic Combat Models with Differential Equations", Report by Vector Research, Inc., Ann Arbor, MI 48106, November 1976, 28 pp. (Abstract in ORSA Bulletin, Vol. 23, Supplement 2, November 1975, p. B-289).

AT Attempts to bound the SL Square Law. Results not correct. Gives an improved method of estimating SL Square Law moments (not assuming absorption probabilities to be zero), same as Farrell and Freedman (1975) except does not include heterogeneous case.

AE IV p. 28. V p. 27.

FARRELL, R.L. and FREEDMAN, R.J. (1975)

"Investigations of the Variation of Combat Model Predictions with Terrain Line of Sight", paper prepared for US Army Materiel Systems Analysis Agency, Aberdeen Proving Grounds, Aberdeen, MD 21005 by Vector Research, Inc., Ann Arbor, MI 48106, January 1976, 116 pp. DTIC# AD B011157.

AT Appendix B, p. 85, gives an improved method of estimating SL Square Law moments (not assuming absorption probabilities to be zero).

GAFARIAN, A.V. and ANCKER, C.J., JR. (1984)

"The Two-on-One Stochastic Duel", NRLQ, Vol. 31, No. 2, June 1984, pp. 309-324.

AE I and II pp. 318-320. V pp. 321-323. (includes GR comparisons)

(Also TR-43-83, U.S. Army TRASANA, White Sands Missile Range, NM 88002, Dec 1983, 58 pp. DTIC# AD A143679. Also ISE TR 83-1, Dept of Industrial and Systems Engineering, Univ of Southern California, Los Angeles, CA 90089, March 1983, 58 pp.)

AE I and II pp. 20-24, I2-11. V pp. 25-29, I12-21. (Includes GR comparisons)

GRAINGER, P.L. (1976)

"The Use of a Stochastic Tank Engagement Model to Examine the Effects of Moving Target Correlation and to Derive Equivalent Lanchester Equations", MS thesis in Operational Research, Dept of Statistics and Operational Research, Brunel University, Uxbridge, UK, June, 1976, 88 pp.

AT Analyzes behavior near t = 0 of models whose overall kill rates are proportional to products of the number remaining on each side raised to integral powers (App. G).

GYE, R. and LEWIS, T. (1974)

"Some New Results Relating to Lanchester's Square Law", Research Report, Department of Mathematical Statistics and Sub-Department of Operational Research, University of Hull, Humberside, UK, January 1974, 25 pp.

AT Incorrectly argues that L Square Law trace is modal trace for SL (p. 7). It is true for transformed spaces. Also gives approximate values of Square Law SL terminal variances for special cases (pp. 12, 14-15).

AE II p. 21. III p. 23,24. IV p. 21. V pp. 15,16.

GYE, R. and LEWIS, T. (1976)

"Lanchester's Equations: Mathematics and the Art of War. A Historical Survey and Some New Results", Mathematical Scientist, Vol. 1, 1976, pp. 107-119.

AT Shows that, for large initial numbers, the terminal SL Square Law distribution is asymptotically Normal (p. 114)

AE I p. 118. II p. 117, 118. IV pp. 114, 117. V p. 115, 117.

HARDECK, W. and HILDEN, H. (1967)

"Eine Stochastiche Erweiterung Der Lanchester-Theorie" ("A Stochastic Extension of Lanchester Theory") Dipl.-Math. Z.O.R. Trier, 55 Trier, Treverer Strabe 1. West Germany, 1967, 10 pp. NATCON. (In German, English translation available).

AT Indirectly shows that difference terms between all possible L functions and equivalent SL mean value functions exist (equations (10), p. 5). Gives explicit differences for the SL Square Law with time-dependent kill probabilities and firing rates.

HARTLEY, D.A., HAGUES, J.N. and KETTLE-WHITE, W. (1982)

"STOCHADE: A Combat Model for Comparing Deterministic and Stochastic Lanchester's Equations", Paper, RMCS, Apr 1982. 27 pp.

AE IV and V p. 10 and Figs. 3-7

(Also "Deterministic and Stochastic Lanchester's Equations: A Comparison Using Simulation Techniques", OR/WP/12, RMCS, Feb 12, 1982).

AE IV and V p. 12-A-11 and Figs. 3-7.

HELMBOLD, R.L. (1966)

"A 'Universal' Attrition Model", OR, Vol. 14, No. 4, July-Aug 1966, pp. 624-635.

(Also Paper, Combat Operations Group, Ft. Belvoir, VA, May 5, 1965, 23 pp.).

AT Non-rigorous development of L equivalents to various models of volley firing at discrete time intervals (pp. 632-635).

ISBELL, J.R. and MARLOW, W.H. (1956)

"Methods of Mathematical Tactics", Logistics Papers, Issue No. 14, Logistics Research Project, George Washington Univ, Washington, DC, Sept 1956, 195 pp.

AT Rederives the result of Brown (1955) given above (pp. I-25 through I-42).

JAMES, B.A.P. (1981)

"A Random Walk Through Lanchester Square", Working Paper 37/7 (2/81), DOAE, MOD, UK, June 1981. 55 pp.

AE I pp. 21-23,31,32,39,40,42,44,46,52. II pp. 21,22,23,28,29, 31,32,35,37,39,40,42,44,46,52,55. III pp. 24,25,33,34,37,39, 40,42,44,46,53,54. IV pp. 33,34,37,39,40,42,44,46,53,54. V pp. 24,25,27,28,29,33,34,35,43,53,54,55.

KARR, A.F. (1975a)

"On Simulations of the Stochastic, Homogeneous, Lanchester Square-Law Process", Paper P-1112, IDA, Sept 1975. 27 pp. IDA Log# HQ75-17242. DTIC# AD A015658.

AE I pp. 6-9, 13. IV pp. 17,18,21,22. V pp. 24,25.

KARR, A.F. (1975b)

"On Simulations of the Stochastic, Homogeneous, Lanchester Linear-Law Attrition Process", Paper P-1113, IDA, Sept 1975, 28 pp. IDA Log# HQ75-17243.

AE I pp. 6-8,10. II p. 10. IV pp. 20-23. V pp. 25,26. VI pp. 12-14,25,26.

KARR, A.F. (1976)

"Deterministic Approximations of Some Stochastic Models of Competing Populations", Technical Report No. 239, Dept of Mathematical Sciences, Johns Hopkins Univ, Baltimore, MD, March 1976, 24 pp.

AT Rigorous proof that transformed versions of the Square Law, Square Law with reinforcements, Linear Law and Mixed Law SL mean value functions converge in probability to transformed versions of the L equations.

AE IV p. 22.

KISI, T. (1965)

"The Lanchester Theory of Combat", Defense Academy Research Faculty OR Course, Japan, 1965, 61 pp. (In Japanese, English translation available).

AE II p. 52. IV pp. 52,54. V p. 52.

KISI, T. (1966)

Private communication to C.J. Ancker, Jr., Oct. 11, 1966, 2 pp.

AT Derives Normal integral approximation to the SL Square Law probability of winning.

KOOPMAN, B.O. (1970)

"A Study of the Logical Basis of Combat Simulation", OR Vol. 18, No. 4, July-Aug 1970, pp. 855-882.

AT Shows that SL Square Law mean value functions tend to L solutions for large initial numbers if absorption probabilities are ignored or discrete variables are non-rigorously treated as continuous (pp. 867-871). A model with detection is similarly treated (pp. 871-879).

LEE, C.Q. (1979)

A Probabilistic Approach to the Prediction of Many-on-Many Combat Outcome; The Marriage of the Classical Ruin Problem and the Lanchester's Second Law", Report 3510-79-204, Northrop Corporation, Hawthorne, CA 90250, July 27, 1979, 13 pp.

AE IV pp. 10-13.

LEE, W.Y. and WANNASILPA, A. (1972)

"Comparison of a Deterministic and a Stochastic Model for the Probability of Winning in a Two-Sided Combat Situation". MS Thesis in Operations Research, NPGS, Sept 1972. 40 pp. DTIC# AD 756536.

AE IV pp. 19-25,27-33.

MARSHALL, C.W. (1965)

"Probabilistic Models in the Theory of Combat", Transactions, New York Academy of Sciences, Series II, Vol. 27, 1965, pp. 477-487.

AT For the Square Law with reinforcements shows that SL tends to L if absorption probabilities are assumed zero and boundary replacements are ignored.

MORSE, P.M. and KIMBALL, G.E. (1956)

Methods of Operations Research, Technology Press, MIT, 1st Ed. Rev., 1956. pp. 67-71.

AE I p. 68. II p. 70. IV pp. 68,70. V p. 70.

PERLA, P.P. and LEHOCZKY, J.P. (1977)

"A New Approach to the Analysis of Stochastic Lanchester Processes. I. Time Evolution". Technical Report No. 135, Dept of Statistics, Carnegie-Mellon University. Pittsburgh, PA 15213. Sept 1977. 34 pp. DTIC# AD A045176.

AT Diffusion approximation to the early part of the SL Square Law battle.

AE I, V and VI pp. 27,28.

SNOW, R.N. (1948)

"Contributions to Lanchester Attrition Theory", Report RA-15078, Project RAND, Douglas Aircraft Company, Inc., Santa Monica, CA, April 5, 1948. 33 pp.

AT Difference terms between the differential equations for the SL mean-value functions and the L functions for a generalized Square Law (pp. 24,25) and second moments for the Square Law (assuming absorption probabilities are zero) (p. 25) are derived.

SPRINGALL, A. (1968)

"Contributions to Lanchester Combat Theory", Ph.D. Thesis in Statistics, Virginia Polytechnic Institute, Blacksburg, VA, Mar 1968. 205 pp. UMic# 68-12,660.

AE I pp. 164-166. II pp. 49,158,159,164,165,166,167. III pp. 54,161, 162,177. IV pp. 49,156,158,159,168,169,177,182. V pp. 49,54,158, 159,170,172,173,182.

TAYLOR, J.G. (1972)

"Applications of Differential Games to Problems of Military Conflict: Tactical Allocation Problems-Part II", Report NPS55TW72111A, NPGS, Nov 1972, 506 pp. DTIC# AD 758 663.

AT First and second order diffusion approximations to SL Square Law distribution (pp. I-42 through I-46).

TAYLOR, J.G. (1983)

Lanchester Models of Warfare. Vol 1, Research Monograph published by the Military Applications Section of the Operations Research Society of America, 1983, 985 pp.

AT Derives relationship between A and B sides' L functions and SL mean value function differences for the Linear Law (equation 4.12.24, p. 505).

TOMPKINS, C. (1953)

"Steps Toward Approximations: Probabilistic Attrition Functions", INA 53-6, Report, National Bureau of Standards, Los Angeles, CA, January 23, 1953, 15 pp.

AT Reproduces some of Snow's (1948) prior results given above.

VENTSEL, Y.S. (1964)

"Introduction to Operations Research", Soviet Radio, Moscow, translated from Russian by the U.S. Air Force Foreign Technology Division, 1964. DTIC# AD B030422L.

AT Invokes the Palm-Khintchine Theorem to justify using SL equations for a GR model (pp. 215-216). This is inappropriate.

WALLIS, P.R. (1967)

"A Model for Force Attrition", AUWE Tech Note 191/65 (second edition) revised, Admiralty Underwater Weapons Establishement, Portland, UK, May 1967, 22 pp.

AE II, IV and V p. 15.

WATSON, R.K. (1976)

"An Application of Martingale Methods to Conflict Models", OR Vol. 24, No. 2, Mar-April 1976. pp. 380-382.

AT Uses Martingale Theory to reproduce Gye and Lewis (1974) results above on terminal variances.

WEALE, T.G. (1971)

"The Mathematics of Battle I. A Bivariate Probability Distribution", M7129, DOAE, MOD, UK, Dec 1971, 38 pp.

AE V and VI pp. 14-23.

WEALE, T.G. (1972)

"The Mathematics of Battle II. The Moments of the Distribution of Battle States", M7130, DOAE, MOD, UK, Oct 1972, 65 pp.

AT Gives an approximate time from initiation of combat during which L approximates mean of SL for the Square Law with any specified accuracy (pp. 19, 20). Reproduces Clark's (1969) results on SL Square Law variances early in the combat (p. 10).

AE V and VI pp. 49, 50.

WEALE, T.G. (1975)

"The Mathematics of Battle V. Homogeneous Battles with General Attrition Functions", M7511, DOAE, MOD, UK, Aug 1975, 83 pp.

AE IV pp. 64,65. V pp. 29-37. VI pp. 29-37,64,65.

WEALE, T.G. (1976)

"The Mathematics of Battle VI. The Distribution of the Duration of Battle", M76126, DOAE, MOD, UK, June 1976, 84 pp.

AE III pp. 15-17, 43-45. IV pp. 17, 45,69,70. V pp. 15-17, 43-45,69,70.

WEISS, G.H. (1963)

"Comparison of a Deterministic and a Stochastic Model for Interaction Between Antagonistic Species", Biometrics, Vol. 19, Dec 1963. pp. 595-602.

AT Give exact expressions for the SL Linear Law conditional terminal distribution and mean. The win probability is given exactly and as an asymptotically Normal integral approximation.

AE II p. 601. IV pp. 599, 600.

WILLARD, D. (1962)

"Lanchester as a Force in History: An Analysis of Land Battles of the Years 1618-1905", Technical Paper RAC-TP-74, Research Analysis Corporation, Bethesda, MD, Nov 1962, 37 pp. DTIC# AD 297375L.

AT Non-rigorous argument that Square Law SL mean value functions converge in probability to L functions. True only for transformed spaces (pp. 31-33).

WILLIAMS, G.T. (1963)

"Stochastic Duels - II", SP-1017/003/00, System Development Corp, Santa Monica, CA, Sept 13, 1963, 61 pp. DTIC# 420 515.

AT Non-rigorous argument that both Linear (pp. 31-47) and Square Law (pp. 47-60) SL mean value functions in time-independent state-space converge in probability to L functions. True only for transformed spaces.

<u>WILLIS, R.F.</u> (1982)

"Stochastic Process Models of Combat", Working Paper OR/WP/28, RMCS, UK., Aug 1982, 27 pp.

AT Derives moments of the SL Square Law for large initial numbers from moment generating function (p. 6). Good only early in the combat (ignores absorption probabilities).

PART TWO - FIGURES AND TABLES FROM THE LITERATURE

0. INTRODUCTION

In this Part we have included almost all the calculated and estimated (usually by simulation) results that we have found in the literature and which are catalogued under AE in the Annotated Bibliography. Considerable effort has been expended in redrawing all figures and retyping all tables in common notation. These have been carefully organized and then cross referenced between the Annotated Bibliography and the figures and tables of Part Two and between the sections within Part Two so that working analysts may readily locate all the details they may be interested in.

The only omitted material is from Craig (1975) pages 71-81, 83, 85-91, 94, 96, and 98-126, and is a number of figures where Square Law L functions and SL mean value functions are plotted in the manner of the figures in Section I following. However, in the Craig plots (all for annihilation situations) all functions have been normalized by dividing them by their initial values. This has, in essence, divided the SL-L difference by their common initial values. For breakpoint situations, the normalizing factors were $a_0 - x(t_f)$ and $b_0 - y(t_f)$. At first glance these appear to be appropriate transformations. In fact, they obscure the absolute magnitude of the difference (by greatly reducing them). Consequently, we have chosen to omit these figures as otherwise it would require that they be recalculated and replotted on an unnormalized basis, which did not appear to be worth the effort.

In many figures and tables from original sources, deterministic values are not given. We have calculated them and included the results to make comparisons possible. Any errors are strictly our responsibility.

A. The Significance of Parity

The parameters for stochastic parity (which we define to be when P(A) = P(B)) will not have the same values as they do for L parity (except for strict parity). Still when one is far from L parity the fire-fights will be decisively lopsided in favor of the stronger side in both models and not be very interesting. This fact is especially important since L parity is easy to calculate and SL parity is usually very difficult to determine. Strict parity is defined for the Square and Linear Laws as $\alpha = \beta$, $a_0 = b_0$ and $a_f = b_f$ and for the Mixed Law (with A linear and B square) as $\alpha = \beta$, $a_0 = b_0^2$, and $a_f = b_f^2$. Non-strict L parity is as follows: for the Linear Law, $\alpha(a_0 - a_f) = \beta(b_0 - b_f)$; for the Square Law, $\alpha(a_0^2 - a_f^2) = \beta(b_0^2 - b_f^2)$; and for the Mixed Law (A Linear), $\alpha(a_0 - a_f) = \beta(b_0^2 - b_f^2)$.

B. Draws

There are some examples in what follows of models where draws are possible outcomes. This matter was not discussed in Part One and deserves some attention here. Let us suppose that each side has a breakpoint (for sake of generality) given by a_f and b_f respectively. Whichever side reaches it's casualty breakpoint first will surrender or break and run. Now, let us further suppose that each side has a point at which it's casualties are such that it no longer has the capability of winning but are not so large as to have reached it's breakpoint. Consequently, it would disengage if the other side would also retire. We shall denote these points by a new notation, a_D and b_D. The time-independent joint probability state space for stochastic models of Figure 4 is reproduced below as Figure 18 without showing all the lattice points which are the possible system states. What we do show are all the subsets of states which have common possible outcomes in the breakpoint and draw situation.

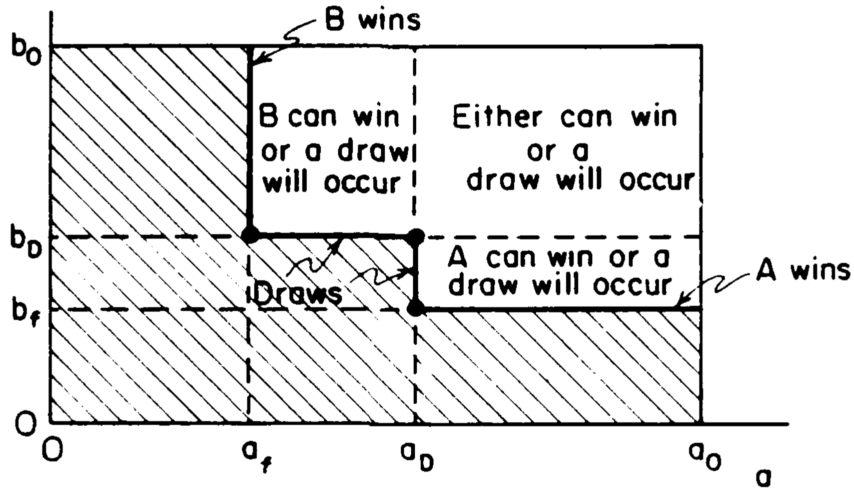


Figure 18. The Stochastic Representation of draws

All shaded areas and the three corners with large dots contain states which cannot occur.

The corresponding deterministic situation is shown in Figure 19, the phase-space of Figure 2(d).

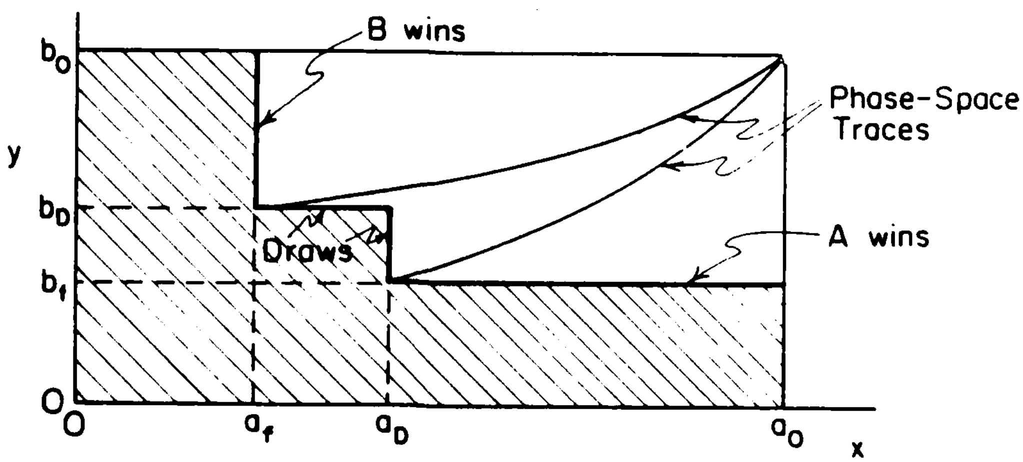


Figure 19. The Deterministic Representation of Draws.

Any phase-space trace lying between the curves shown will surely end in a draw; a trace below both will result in an A win and one above both will be a B win.

C. Special Models

There is no dicusssion in Part One of two models which are in the Annotated Bibliography and whose examples are in this part.

1. The Weale Special Model

This model differs from the others in that attrition is of the form $\alpha = a(c_1 + c_2 b)$ and $\beta = b(c_3 + c_4 a)$, where c_1, c_2, c_3, and c_4 are constant attrition coefficients. This model contains as special cases the Linear, Square and Mixed Laws.

2. The Clark LORSUM Model

This model is considerably more complicated than any of the others. Briefly, it involves the following complications:

(1) Several different groups on each side with individual group characteristics.

(2) Different kill rates by each group against every opposing group.

(3) A rate at which each group detects each opposing group.

(4) A time delay from detection to acquisition for all on each side. This is called a shift coefficient.

(5) Acquisition autocorrelation between units internal to each group.

These factors are displayed in Table I-7 for particular examples. The details of this model are considerably more involved that is discussed above, but we shall not elaborate further here.

I. THE DIFFERENCE BETWEEN SL MEAN VALUE FUNCTIONS AND L FUNCTIONS

All figures and tables in this section display the difference except Tables I-2, I-3(a), I-6 and Figure I-18, which do not have the deterministic information. They are included for completeness.

Additional information on the differences is given in Figures VI-3 through VI-12.

Parity may be observed in Figures I-1(b), I-3 (solid curves only), I-5(c),(d), I-21(b), I-24(c), I25(a),(c),(e), I-26(c),(d), VI-3 and VI-4 while strict parity occurs in Figures I-1(a),(c), I-4, I-9, and I-12. There are scattered parity points in various tables but being at isolated time points they are of little interest. As observed earlier, at or near parity L is particularly poor as an approximation to SL.

An important point about breakpoints is illustrated in Figures I-3, I-4, and I-5. That is, for L the breakpoint curves are identical with the non-breakpoint curves until the breakpoints are reached, at which point the L curves are horizontal straight lines. However, the SL breakpoint curves are higher than their non-breakpoint counterparts and depart from the L curves much earlier in time. Contrary to some speculation, breakpoints do not improve the L approximation to SL mean value functions; the effect of absorption probabilities simply occurs earlier.

In Figures I-19 through I-26 we see a two versus one model illustrate the differences in L, SL, and GR mean value functions. One striking point is that for the same value of $\alpha = (p_A / \mu_A)$ various values of P_A greatly change the GR curve. In other words, combining the parameters p_A and μ_A can be grossly inadequate.

The other points mentioned on pages 15 and 16 are well illustrated by these results, and they emphasize that:

(1) L functions are generally inadequate as an approximation to SL mean value functions, especially in the most interesting cases where one side does not have a lopsided preponderance of force.

(2) SL mean value functions are generally inadequate as an approximation to their GR equivalents (at least for the small fire-fights so far considered).

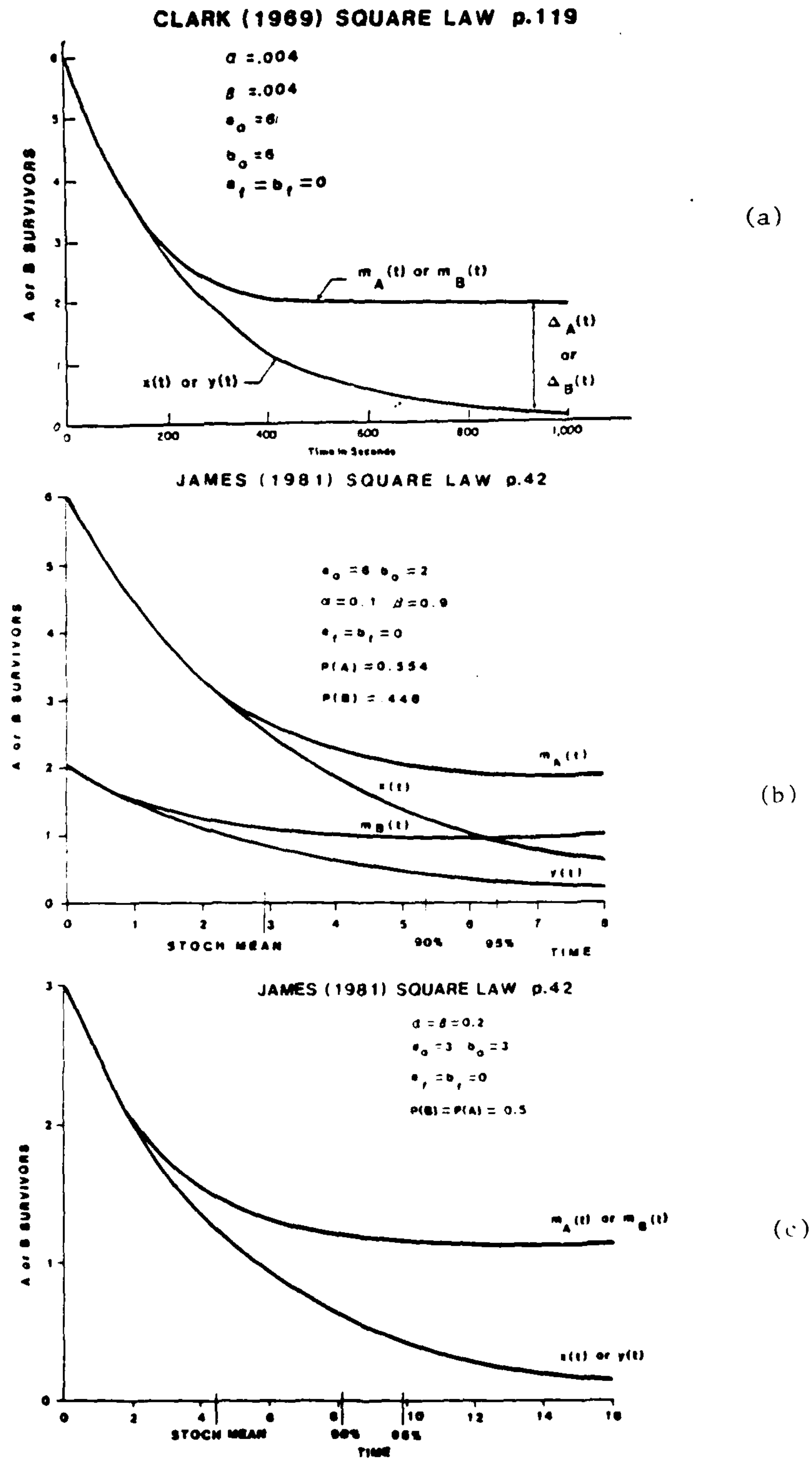
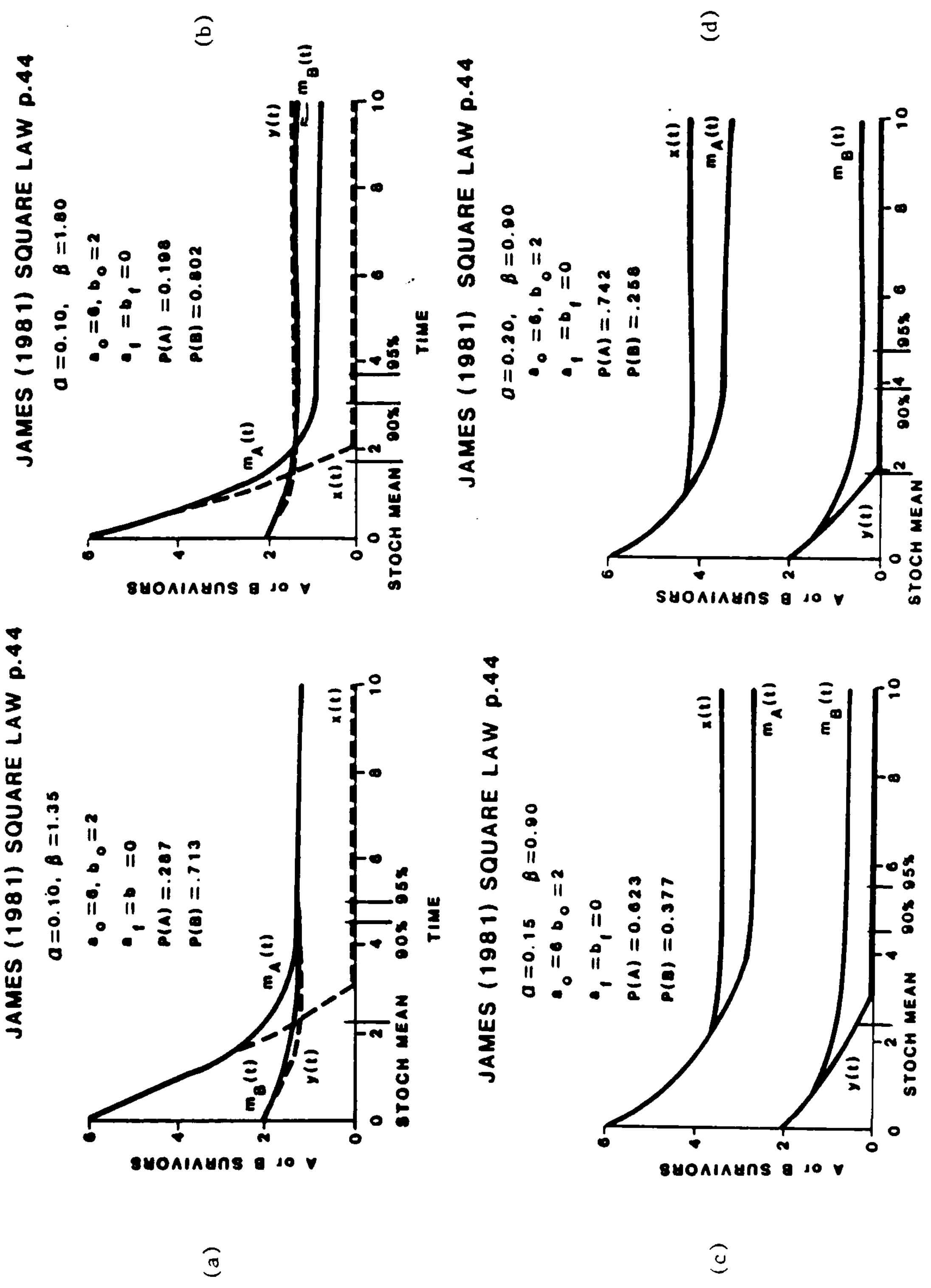
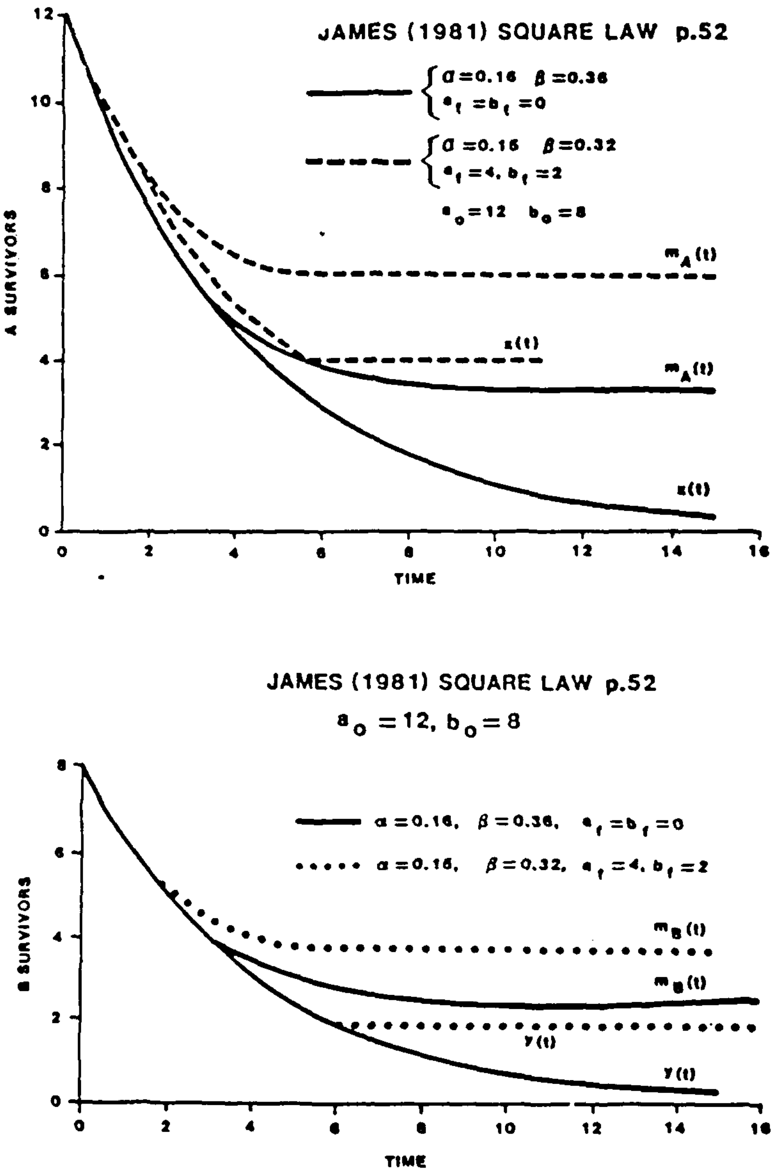

CLARK (1969) SQUARE LAW p.119
α =.004
β =.004
a_o =6
b_o =6
$a_f = b_f = 0$
$m_A(t)$ or $m_B(t)$
$\Delta_A(t)$
or
$\Delta_B(t)$
x(t) or y(t)
A or B SURVIVORS
Time in Seconds
(a)
JAMES (1981) SQUARE LAW p.42
a_o = 6 b_o = 2
α = 0.1 β = 0.9
$a_f = b_f = 0$
P(A) = 0.554
P(B) = .446
$m_A(t)$
x(t)
$m_B(t)$
y(t)
A or B SURVIVORS
STOCH MEAN
90%
95%
TIME
(b)
JAMES (1981) SQUARE LAW p.42
α = β = 0.2
a_o = 3 b_o = 3
$a_f = b_f = 0$
P(B) = P(A) = 0.5
$m_A(t)$ or $m_B(t)$
x(t) or y(t)
A or B SURVIVORS
STOCH MEAN
90%
95%
TIME
(c)

Figure I-1.



JAMES (1981) SQUARE LAW p.44
α=0.10, β=1.35
a_0=6, b_0=2
a_f=b =0
P(A)=.287
P(B)=.713
A or B SURVIVORS
STOCH MEAN
90% 95%
TIME
(a)
JAMES (1981) SQUARE LAW p.44
α=0.10, β=1.80
a_0=6, b_0=2
a_f=b_f=0
P(A)=0.198
P(B)=0.802
A or B SURVIVORS
STOCH MEAN
90% 95%
TIME
(b)
JAMES (1981) SQUARE LAW p.44
α=0.15 β=0.90
a_0=6 b_0=2
a_f=b_f=0
P(A)=0.623
P(B)=0.377
A or B SURVIVORS
STOCH MEAN
90% 95%
TIME
(c)
JAMES (1981) SQUARE LAW p.44
α=0.20, β=0.90
a_0=6, b_0=2
a_f=b_f=0
P(A)=.742
P(B)=.258
A or B SURVIVORS
STOCH MEAN
90% 95%
TIME
(d)

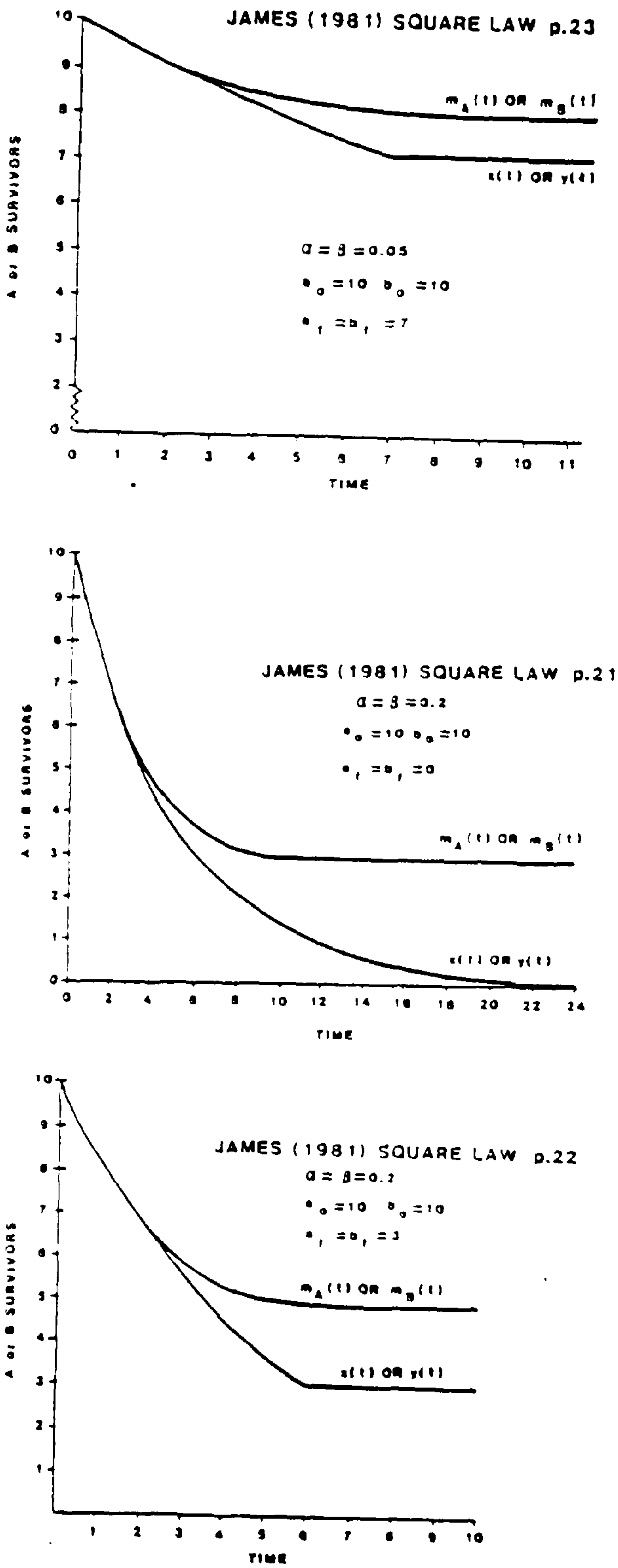
Figure I-2.


JAMES (1981) SQUARE LAW p.52
α=0.16 β=0.36
$a_f = b_f = 0$
α=0.16 β=0.32
$a_f = 4$, $b_f = 2$
$a_0 = 12$ $b_0 = 8$
$m_A(t)$
x(t)
$m_A(t)$
x(t)
A SURVIVORS
TIME
JAMES (1981) SQUARE LAW p.52
$a_0 = 12$, $b_0 = 8$
α=0.16, β=0.36, $a_f = b_f = 0$
α=0.16, β=0.32, $a_f = 4$, $b_f = 2$
$m_B(t)$
$m_B(t)$
y(t)
y(t)
B SURVIVORS
TIME

(a)

(b)

Figure I-3

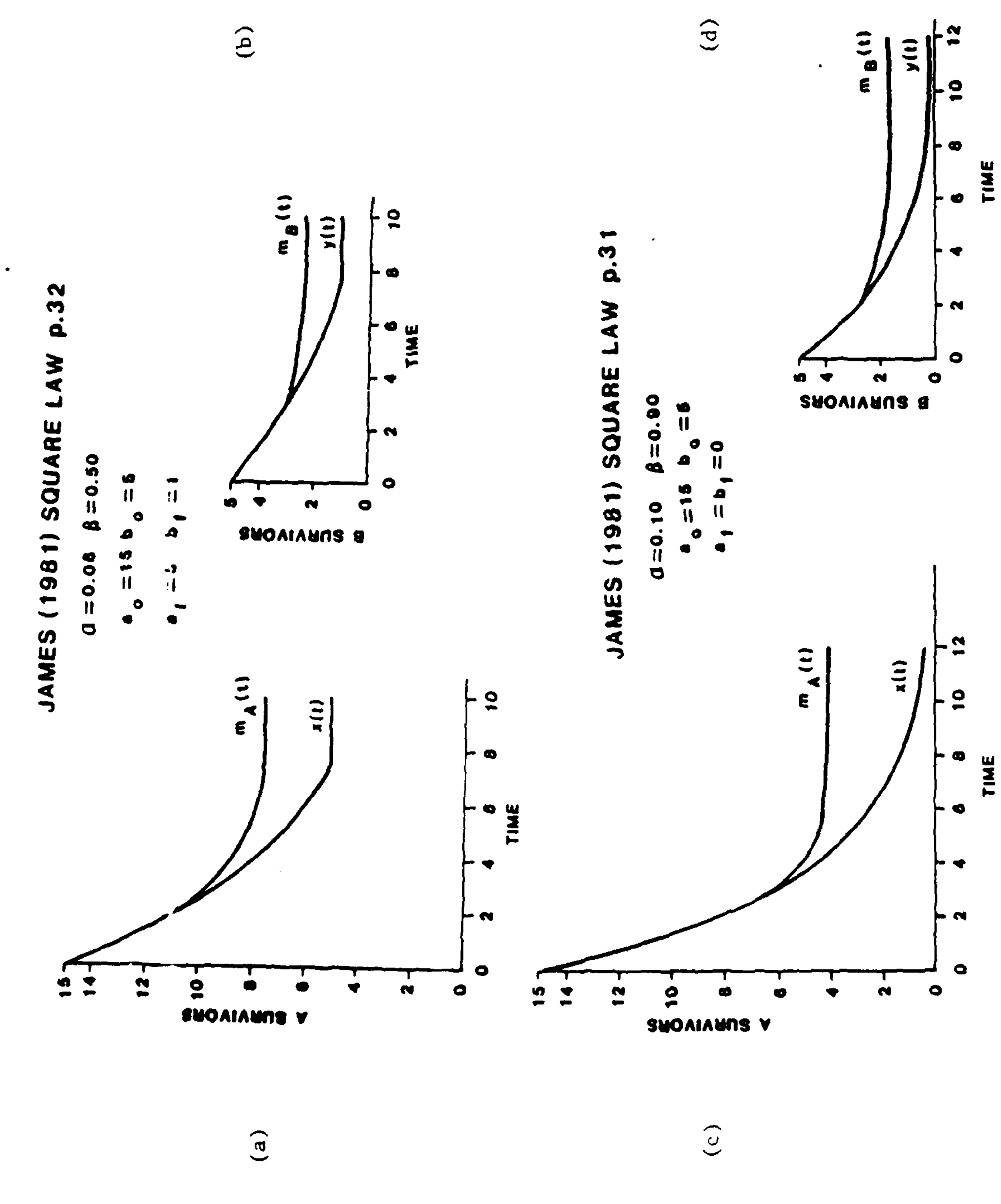


(a)

(b)

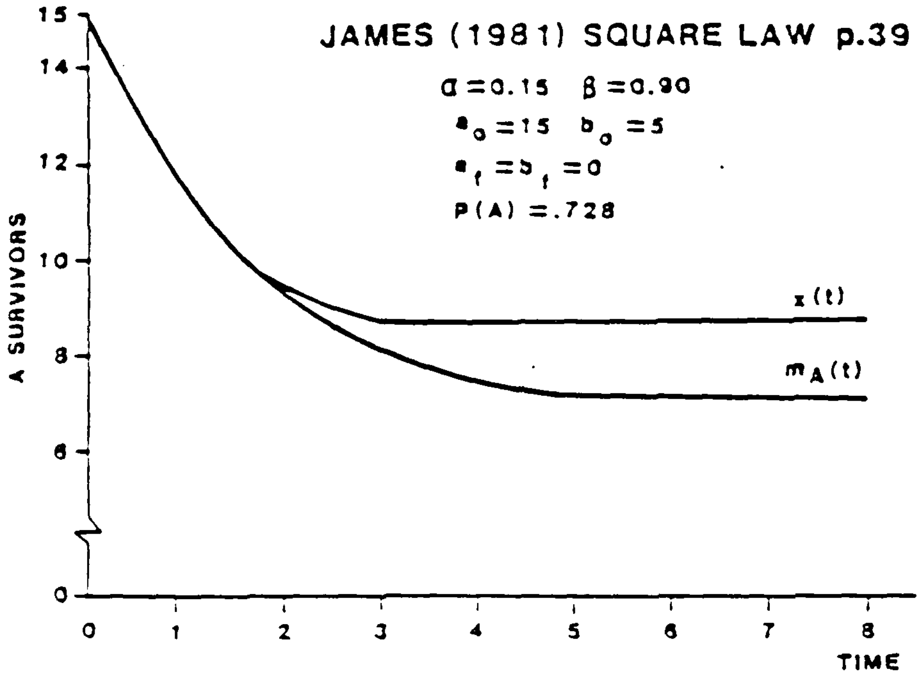
(c)

Figure I-4



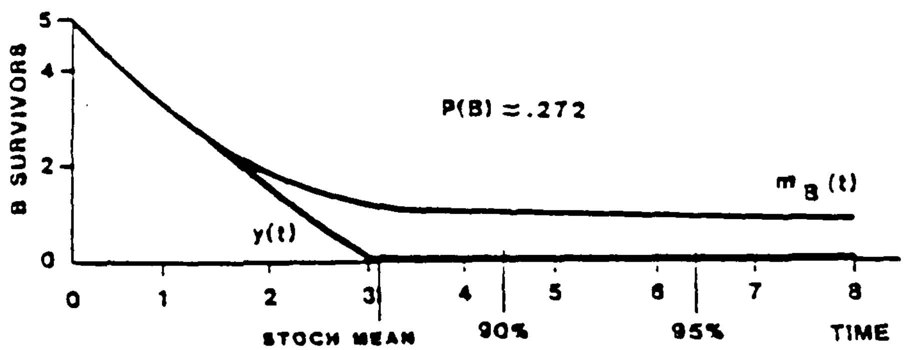
JAMES (1981) SQUARE LAW p.32
$\alpha=0.08$ $\beta=0.50$
$a_o=15$ $b_o=5$
A SURVIVORS
B SURVIVORS
TIME
$m_A(t)$
$x(t)$
$m_B(t)$
$y(t)$
(a)
(b)
JAMES (1981) SQUARE LAW p.31
$\alpha=0.10$ $\beta=0.90$
$a_o=15$ $b_o=5$
$a_1=b_1=0$
(c)
(d)

Figure I-5.



JAMES (1981) SQUARE LAW p.39
α =0.15 β =0.90
a0 =15 b0 =5
af =bf =0
P(A) =.728
A SURVIVORS
x(t)
mA(t)
TIME

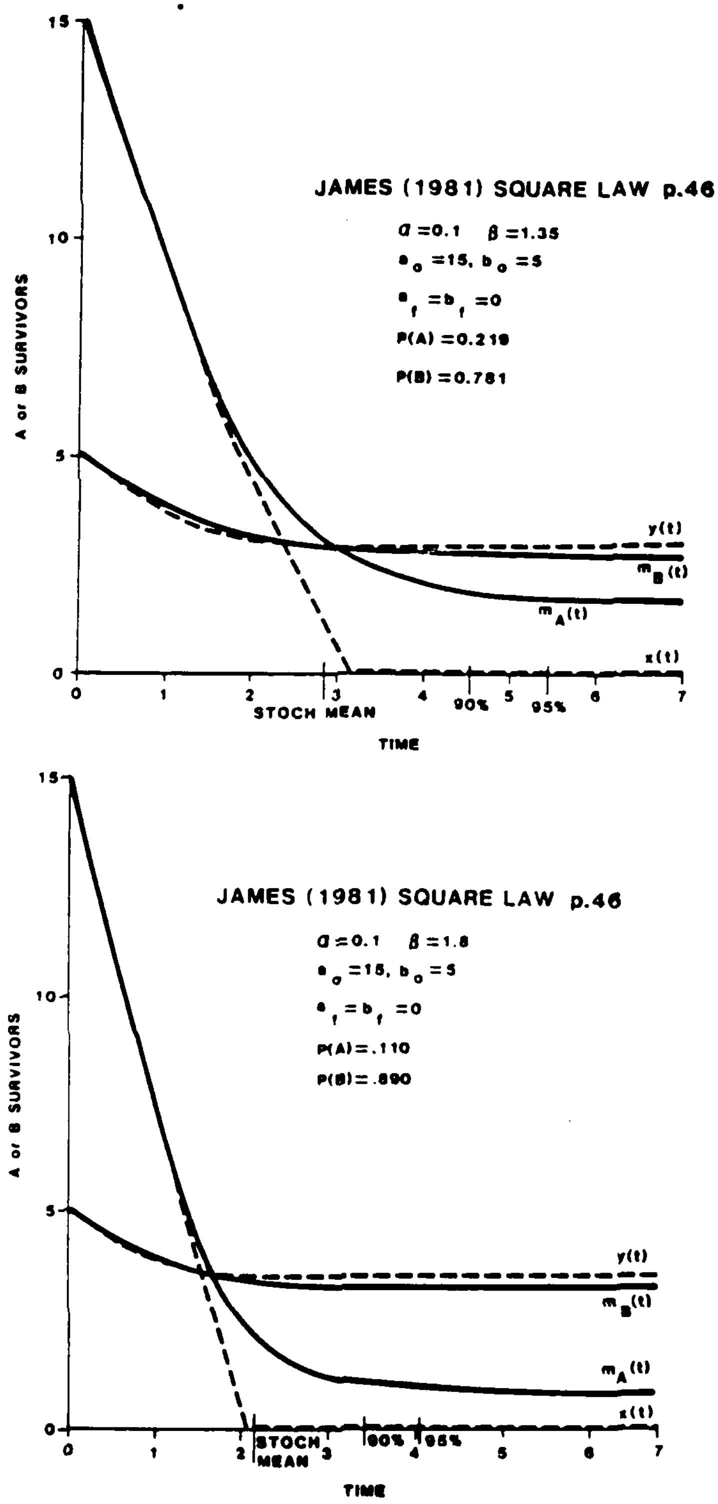
(a)



B SURVIVORS
P(B) =.272
mB(t)
y(t)
STOCH MEAN
90%
95%
TIME

(b)

Figure I-6


JAMES (1981) SQUARE LAW p.46
α =0.1 β =1.35
a_o =15, b_o =5
a_f =b_f =0
P(A) =0.219
P(B) =0.781
y(t)
m_B(t)
m_A(t)
x(t)
A or B SURVIVORS
STOCH MEAN
90%
95%
TIME
JAMES (1981) SQUARE LAW p.46
α =0.1 β =1.8
a_o =15, b_o =5
a_f =b_f =0
P(A)=.110
P(B)=.890
y(t)
m_B(t)
m_A(t)
x(t)
A or B SURVIVORS
STOCH MEAN
90%
95%
TIME

(a)

(b)

Figure I-7.

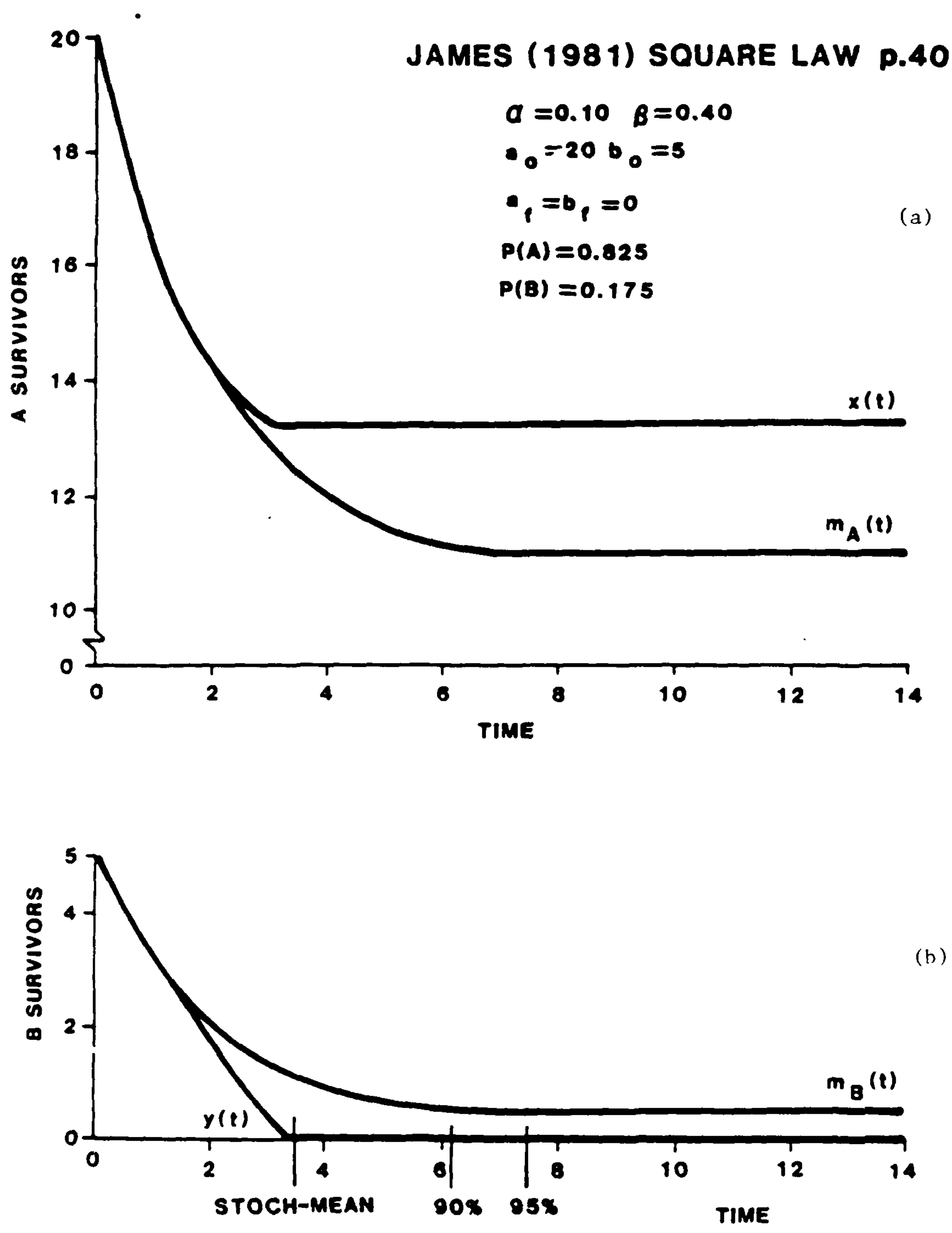
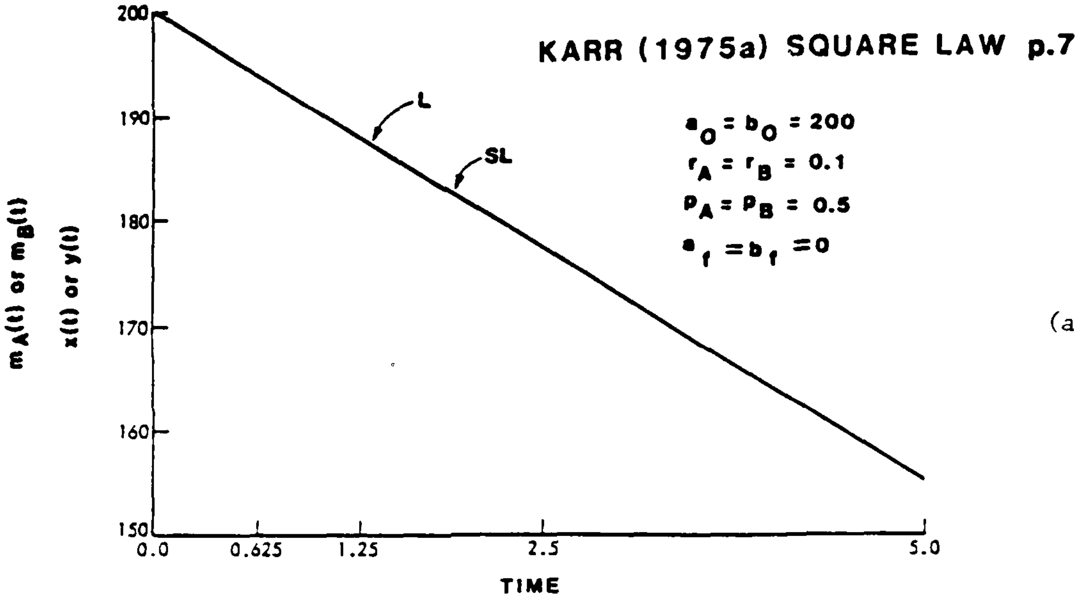
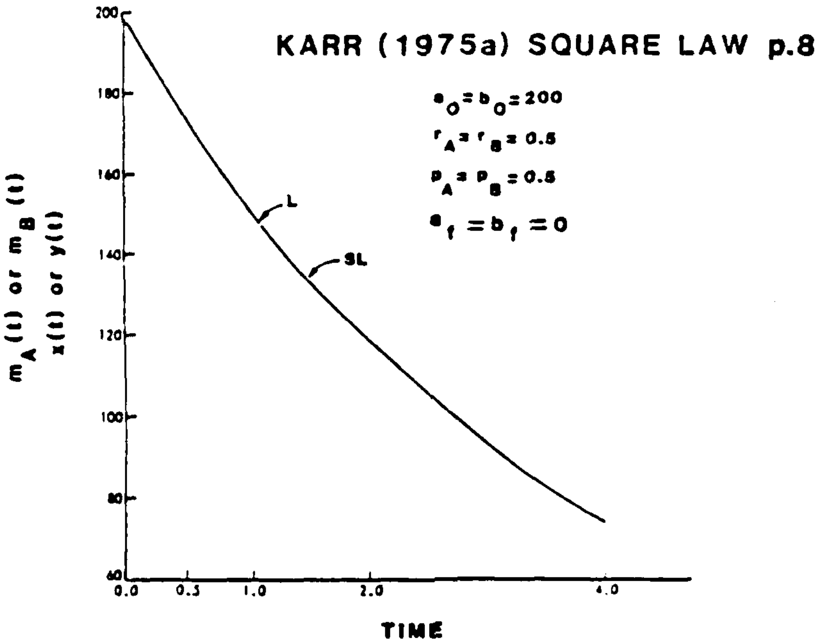

JAMES (1981) SQUARE LAW p.40
α =0.10 β=0.40
a_o =20 b_o =5
a_f =b_f =0
P(A)=0.825
P(B) =0.175
(a)
A SURVIVORS
20
18
16
14
12
10
0
x(t)
m_A(t)
0
2
4
6
8
10
12
14
TIME
B SURVIVORS
5
4
2
0
(b)
m_B(t)
y(t)
STOCH-MEAN
90%
95%
TIME

Figure I-8.


KARR (1975a) SQUARE LAW p.7
$a_0 = b_0 = 200$
$r_A = r_B = 0.1$
$p_A = p_B = 0.5$
$a_f = b_f = 0$
L
SL
$m_A(t)$ or $m_B(t)$
$x(t)$ or $y(t)$
TIME
200
190
180
170
160
150
0.0
0.625
1.25
2.5
5.0

(a)


KARR (1975a) SQUARE LAW p.8
$a_0 = b_0 = 200$
$r_A = r_B = 0.5$
$p_A = p_B = 0.5$
$a_f = b_f = 0$
L
SL
$m_A(t)$ or $m_B(t)$
$x(t)$ or $y(t)$
TIME
200
180
160
140
120
100
80
60
0.0
0.5
1.0
2.0
4.0

(b)

Figure I-9

GYE AND LEWIS (1976) p.118

SQUARE LAW

$a_0 = 23,\ b_0 = 32$

$a_f = b_f = 0$

$\alpha = \beta$

(a)

TIME	$m_A(t)$, $m_B(t)$	L POINT WITH SAME X	TIME	$m_A(t)$, $m_B(t)$	L POINT WITH SAME X
3.1	22, 32	22, 31.3	46.9	10, 25	10, 24.4
9.7	20, 30	20, 29.9	55.2	8, 24	8, 23.6
16.5	18, 29	18, 28.6	63.9	6, 24	6, 23.0
23.7	16, 28	16, 27.4	72.6	4, 23	4, 22.6
31.1	14, 27	14, 26.3	81.6	2, 23	2, 22.3
38.9	12, 26	12, 25.3			

CLARK (1969) p.122

SQUARE LAW

$a_f = b_f = 0$

(b)

a_0	b_0	α	β	t	$m_A(t)$	$m_B(t)$	x(t)	y(t)	$\Delta_A(t)$	$\Delta_B(t)$
6	6	.001	.004	275.	1.059	5.134	0.000	5.195	1.059	-.061
6	6	.0015	.004	291.	1.133	4.640	0.000	4.740	1.133	-.100
6	6	.002	.004	312.	1.227	4.102	0.000	4.240	1.227	-.138
6	6	.004	.004	2500.	1.929	1.929	0.000	0.000	1.929	1.929
8	8	.001	.004	275.	1.226	6.863	0.000	6.922	1.226	-.059
8	8	.0015	.004	291.	1.314	6.213	0.000	6.320	1.314	-.107
8	8	.002	.004	312.	1.426	5.492	0.000	5.623	1.426	-.136
8	8	.004	.004	2500.	2.407	2.407	0.000	0.000	2.407	2.407
12	6	.001	.004	2500.	3.297	1.969	0.081	0.040	3.216	1.929
12	6	.0015	.004	493.	6.474	1.339	6.920	0.000	-.446	1.339
12	6	.002	.004	312.	8.160	1.176	8.480	0.000	-.320	1.176
12	6	.004	.004	137.	10.275	1.060	10.390	0.000	-.115	1.060

Table I-1.

PERLA AND LEHOCZKY (1977) p.27

SQUARE LAW

$a_f = b_f = 0$

$\alpha = \beta = .05$

$t = 15$

(a_0,b_0)	$m_A(t)$ S	$m_A(t)$ D	$\|S-D\|/S$	$m_B(t)$ S	$m_B(t)$ D	$\|S-D\|/S$
(20,20)	9.48 (.041)	9.45	.0032	9.40 (.002)	9.45	.0053
(25,25)	11.85 (.069)	11.81	.0034	11.87 (.061)	11.81	.0051
(30,30)	14.21 (.041)	14.17	.0028	14.16 (.011)	14.17	.0007
(40,40)	18.98 (.190)	18.89	.0047	18.93 (.069)	18.89	.0021
(50,50)	23.64 (.118)	23.62	.0008	23.58 (.251)	23.62	.0017

S = Simulation, 6000 replications. Numbers in parentheses are standard deviations of the S estimates.

D = Perla & Lehoczky (1977) diffusion approximation.

(a)

PERLA AND LEHOCZKY (1977) p.28

SQUARE LAW

$a_f = b_f = 0$

$\alpha = .075, \beta = .030$

$t = 10$

(a_0,b_0)	$m_A(t)$ S	$m_A(t)$ D	$\|S-D\|/S$	$m_B(t)$ S	$m_B(t)$ D	$\|S-D\|/S$
(50,20)	40.18 (.003)	40.16	.0005	6.78 (.056)	6.72	.0088
(75,30)	60.32 (.029)	60.24	.0013	10.08 (.026)	10.09	.0010
(100,40)	80.34 (.066)	80.32	.0002	13.45 (.155)	13.45	.0000
(125,50)	100.32 (.030)	100.41	.0009	16.86 (.089)	16.81	.0030
(250,100)	200.96 (.149)	200.81	.0007	33.70 (.008)	33.62	.0024

S = Simulation, 6000 replications. Numbers in parentheses are standard deviations of the S estimates.

D = Perla & Lehoczky (1977) diffusion approximation.

(b)

Table I-2.

KARR (1975a) pp.6,13

SQUARE LAW

$a_f = b_f = 0$

a_o	b_o	r_A	r_B	p_A	p_B	t	$m_A(t)$	$m_B(t)$
200	200	0.1	0.1	0.5	0.5	0.625 1.25 2.5 5.0	194.40 187.28 176.34 155.86	193.78 187.56 176.80 155.68
200	200	0.5	0.5	0.5	0.5	0.5 1.0 2.0 4.0	176.00 155.62 122.10 73.04	176.12 156.44 121.16 73.54
150	100	0.1	0.225	0.5	0.5	1.25 2.5 5.0	136.38 122.54 103.80	90.98 82.88 63.46
200	200	0.4	0.5	0.5	0.5	1.25 2.5 5.0	145.60 101.76 34.84	155.98 127.22 94.02

(a)

KARR (1975a) pp.9,13

SQUARE LAW

$a_o = 150,\ b_o = 100$

$a_f = b_f = 0$

$p_A = p_B = 0.5$

$r_A = .1,\ r_B = .225$

t	$m_A(t)$	$\frac{m_A(t)}{a_o}$; $(\frac{x(t)}{a_o})$	$m_B(t)$	$\frac{m_B(t)}{b_o}$; $(\frac{y(t)}{b_o})$
1.25	136.38	0.909 (.952)	90.98	0.910 (.952)
2.50	122.54	0.817 (.837)	82.88	0.829 (.837)
5.00	103.80	0.692 (.692)	68.46	0.685 (.692)
Simulation results.				

(b)

Table I-3

CLARK (1969) p.153

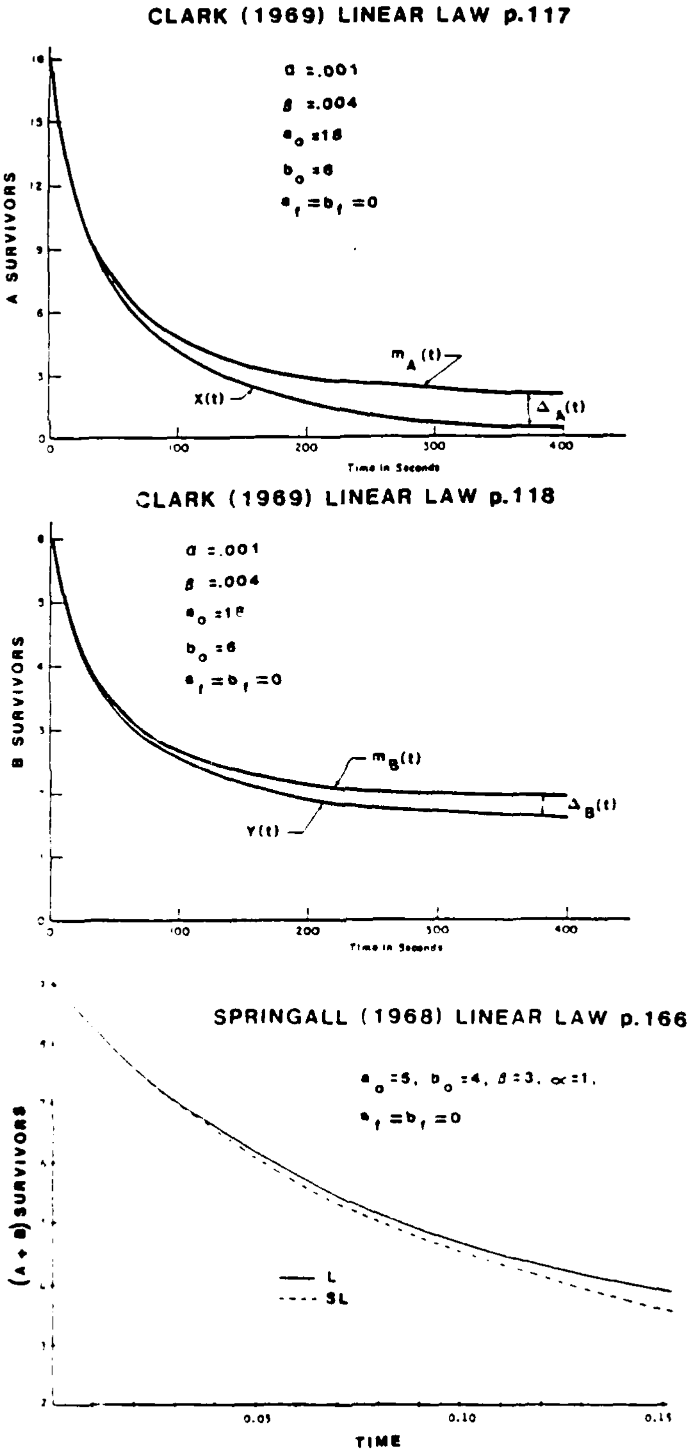
SQUARE LAW (WITH STOCHASTIC ACQUISITION)

$a_f = b_f = 0$

a_0	b_0	α	β	t	$m_A(t)$	$m_B(t)$	x(t)	y(t)	$\Delta_A(t)$	$\Delta_B(t)$
6	6	.001	.004	5000.	.030	4.672	0.000	4.729	.030	-.057
6	6	.0015	.004	5000.	.128	3.987	0.000	4.048	.128	-.061
6	6	.002	.004	5000.	.312	3.338	0.000	3.332	.312	.006
6	6	.004	.004	5000.	1.489	1.489	.316	.316	-1.173	1.173
8	8	.001	.004	5000.	.013	6.304	0.000	6.383	.013	-.079
8	8	.0015	.004	5000.	.085	5.390	0.000	5.504	.085	-.114
8	8	.002	.004	5000.	.264	4.486	0.000	4.568	.264	-.082
8	8	.004	.004	5000.	1.808	1.808	.323	.323	1.485	1.485
12	6	.001	.004	5000.	.961	2.700	.003	2.571	.958	.129
12	6	.0015	.004	5000.	2.528	1.504	.408	.608	2.120	.896
12	6	.002	.004	5000.	4.154	.7[illegible]	3.364	.009	.790	.785
12	6	.004	.004	5000.	8.027	.061	8.268	0.000	-.241	.061

Acquisition Probabilities - (p = q = .85).

Table I-4.

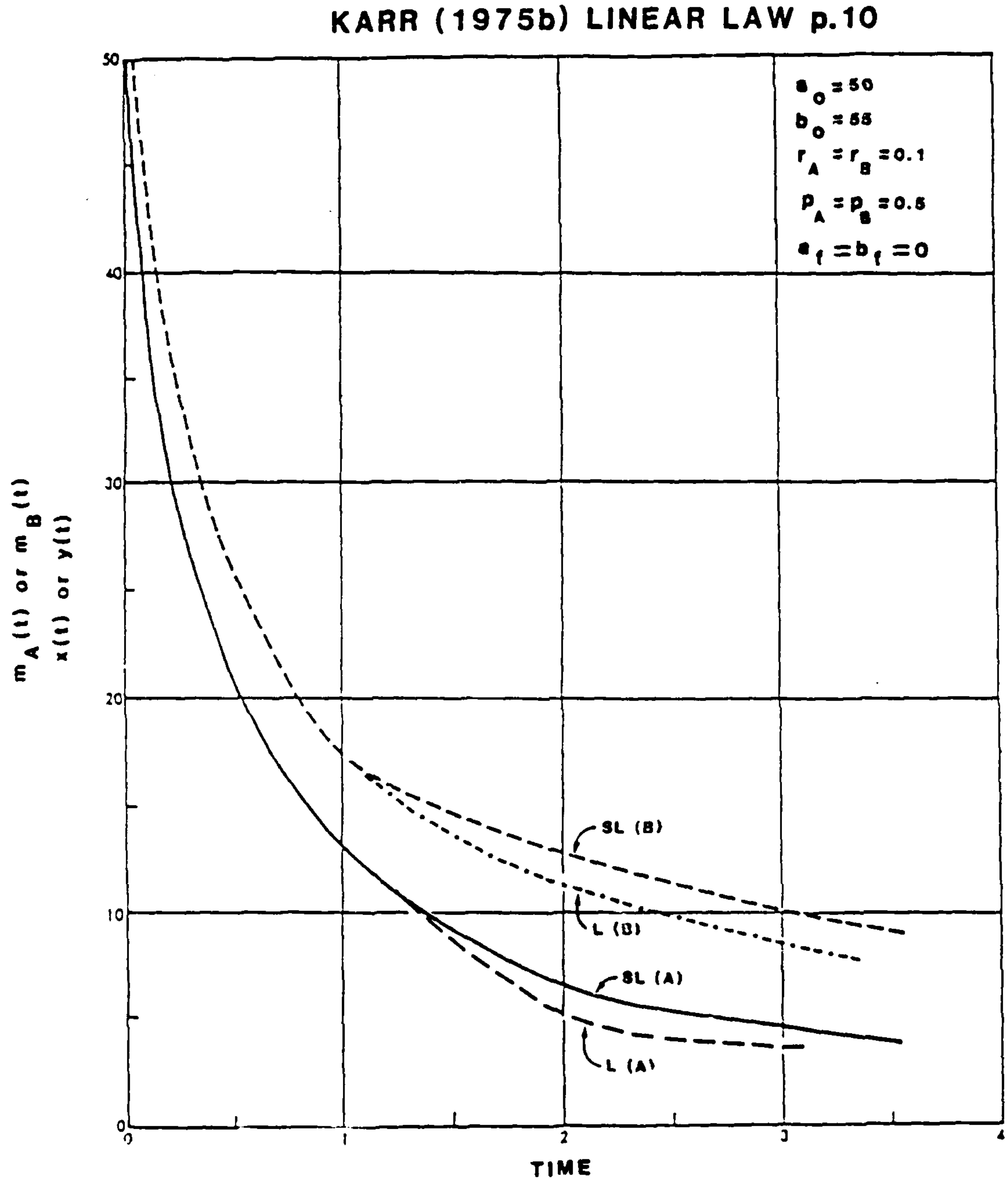


(a)

(b)

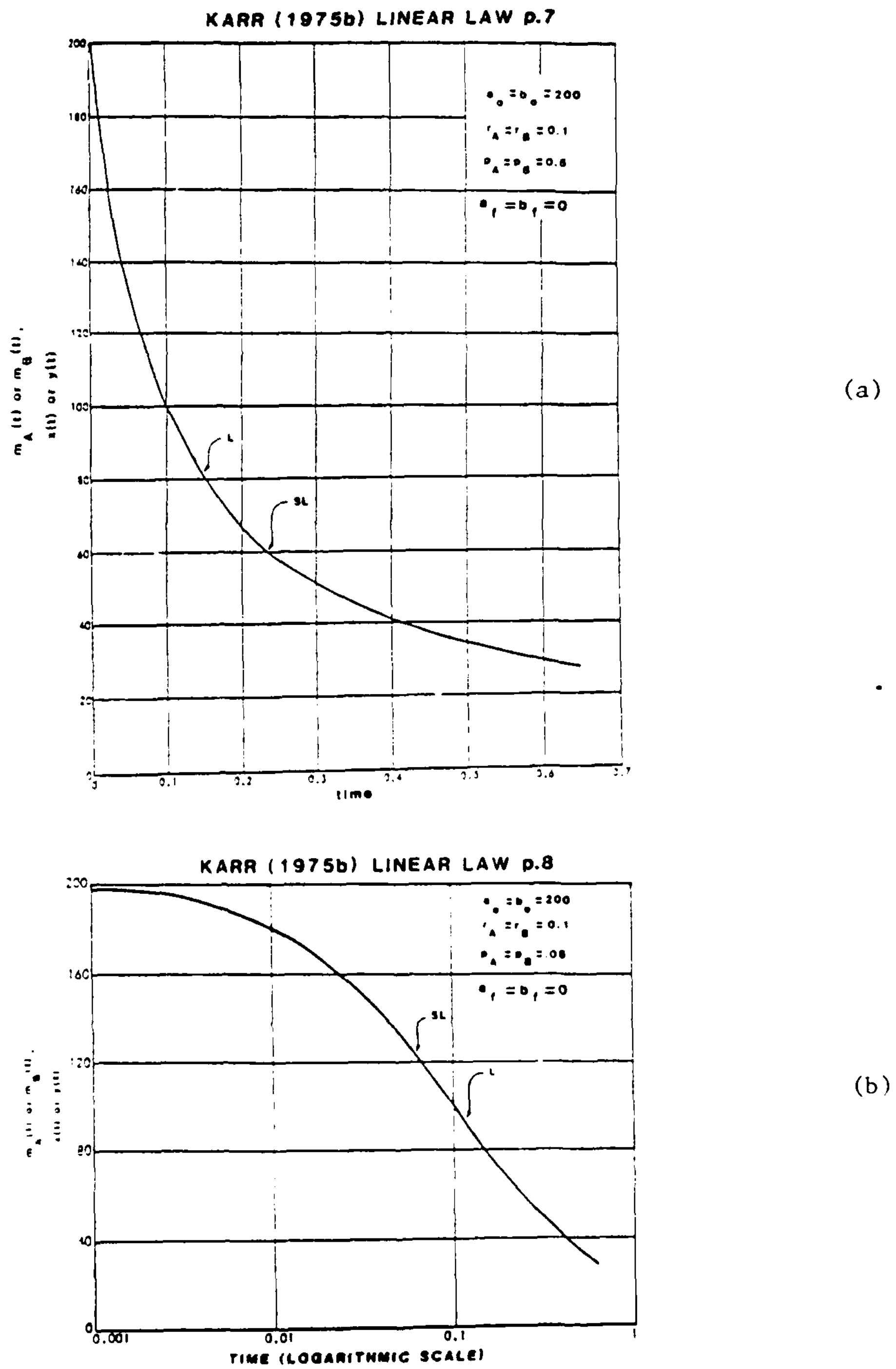
(c)

Figure I-10.



KARR (1975b) LINEAR LAW p.10
$a_o = 50$
$b_o = 55$
$r_A = r_B = 0.1$
$p_A = p_B = 0.5$
$a_f = b_f = 0$
SL (B)
L (B)
SL (A)
L (A)
$m_A(t)$ or $m_B(t)$
$x(t)$ or $y(t)$
TIME
0
10
20
30
40
50
1
2
3
4

Figure I-11.



KARR (1975b) LINEAR LAW p.7
time
L
SL
KARR (1975b) LINEAR LAW p.8
TIME (LOGARITHMIC SCALE)
SL
L
(a)
(b)

Figure I-12.

MORSE AND KIMBALL (1956) p.68

LINEAR LAW

$a_0 = 5,\ b_0 = 3$

$a_f = b_f = 0$

$\alpha = \beta$

k	0	1	2	3	4	5	6	7+
m_A	5.0	4.5	4.0	3.5	3.06	2.72	2.48	2.367
x	5.0	4.5	4.0	3.5	3.0	2.5	2.0	2.0
m_B	3.0	2.5	2.0	1.5	1.06	0.72	0.48	0.367
y	3.0	2.5	2.0	1.5	1.0	0.5	0	0

k = number of kills.

P(B) = (29/128) = 0.2266.

If the B's win, the expected number of B survivors = 1.621.

P(A) = (99/128) = 0.7734

If the A's win, the expected number of A survivors = 3.061.

(a)

CLARK (1969) p.121

LINEAR LAW

$a_f = b_f = 0$

a_0	b_0	α	β	t	$m_A(t)$	$m_B(t)$	x(t)	y(t)	$\Delta_A(t)$	$\Delta_B(t)$
6	6	.001	.004	900.	.024	4.506	0.000	4.500	.024	.006
6	6	.0015	.004	1075.	.107	3.790	0.000	3.750	.107	.040
6	6	.002	.004	1275.	.269	3.134	0.000	3.000	.269	.134
6	6	.004	.004	2500.	1.350	1.350	.093	.093	1.257	1.257
8	8	.001	.004	400.	.020	6.005	0.000	6.000	.020	.005
8	8	.0015	.004	350.	.107	5.040	.005	5.002	.102	.035
8	8	.002	.004	1000.	.214	4.107	0.000	4.000	.214	.107
8	8	.004	.004	2500.	1.571	1.571	.099	.099	1.472	1.472
12	6	.001	.004	850.	.374	3.094	0.000	3.000	.374	.094
12	6	.0015	.004	1500.	1.221	1.958	0.000	1.500	1.221	.458
12	6	.002	.004	2500.	2.351	1.175	.197	.098	2.154	1.077
12	6	.004	.004	375.	6.149	.149	6.000	0.000	.149	.149

(b)

Table I-5.

KARR (1975b) p.6

LINEAR LAW (SIMULATION)

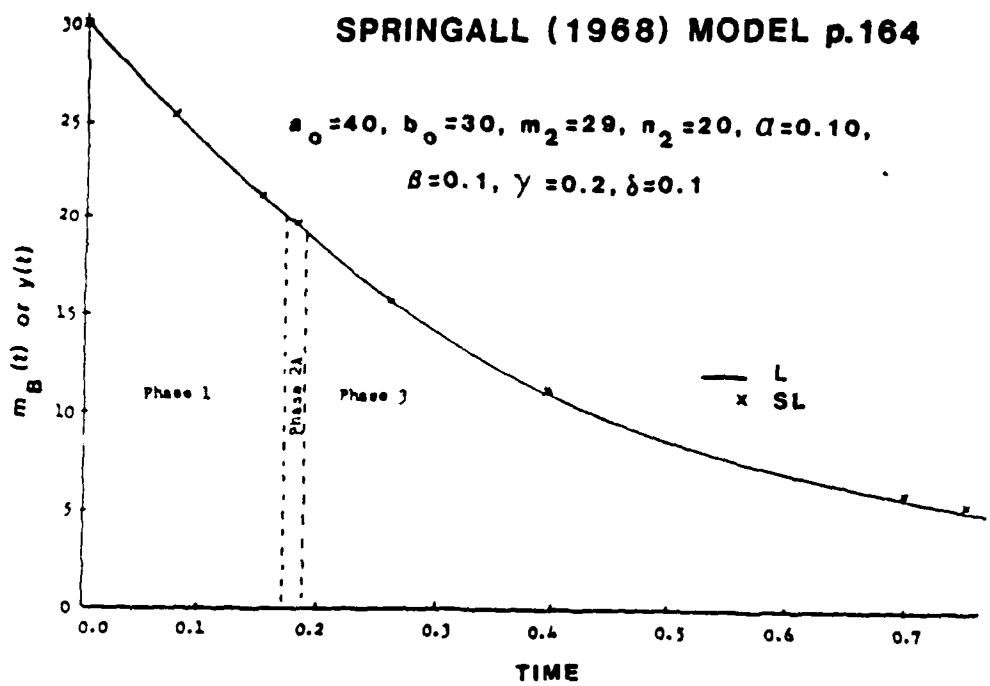
$a_o = 50, \; b_o = 60$

$r_A = r_B = 0.1, \; p_A = p_B = 0.5$

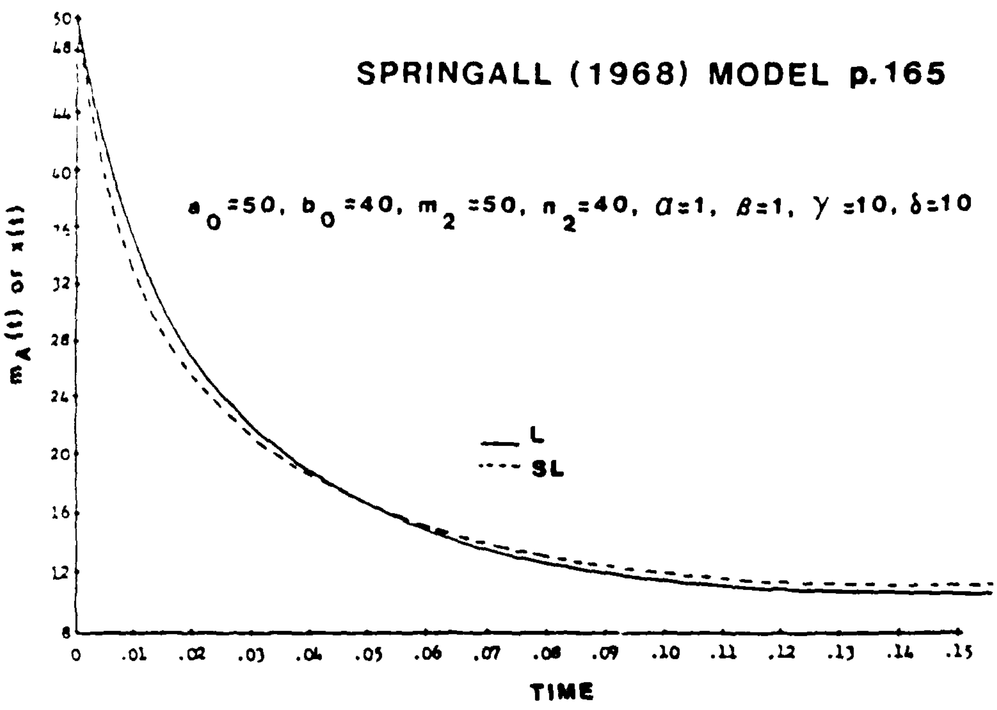
$a_f = b_f = 0$

t	$m_A(t)$	$m_B(t)$
0.000	50.00	60.00
1.000	10.60	19.86
2.000	4.96	14.72
3.000	3.18	13.04
4.000	2.44	11.62
0.000	50.00	60.00
0.100	38.56	48.18
0.200	30.02	41.52
0.300	25.18	35.84
0.400	22.24	30.14
0.500	18.48	28.10
0.000	50.00	60.00
0.010	48.32	58.58
0.020	47.08	57.18
0.030	45.98	56.04
0.040	44.78	55.22
0.050	43.58	53.02
0.060	42.54	51.72
0.070	41.20	51.74
0.080	39.98	50.20
0.090	39.20	49.42
0.100	38.76	47.54
0.000	50.00	60.00
0.001	49.84	59.80
0.002	49.76	59.68
0.003	49.64	59.62
0.004	49.40	59.44
0.005	49.34	59.36
0.006	49.18	59.16
0.007	48.82	58.88
0.008	48.80	58.74
0.009	48.84	58.76
0.010	48.72	58.58

Table I-6



(a)



(b)

Figure I-13.

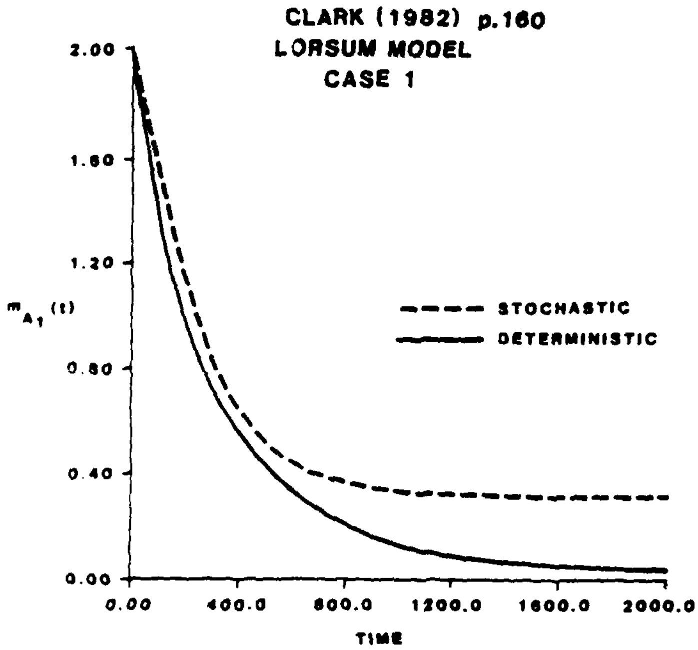
CLARK (1982) pp.157,158

LORSUM MODEL

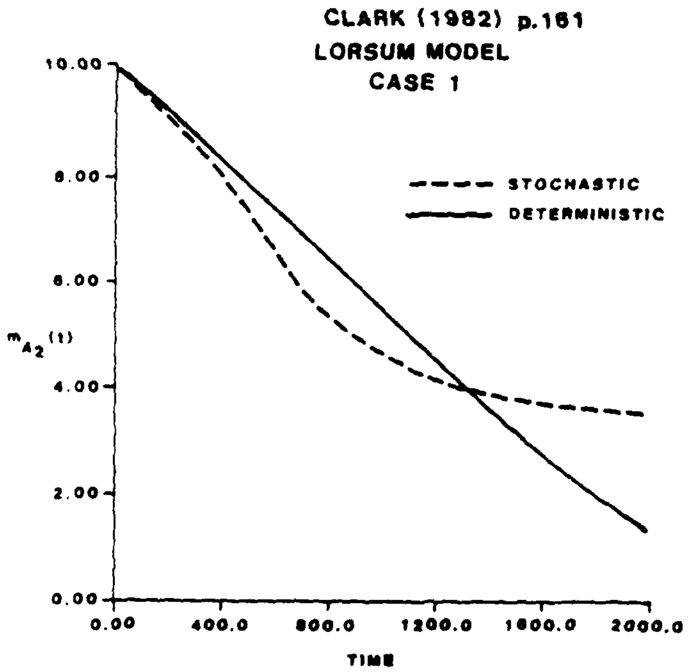
This table defines the parameters for following Figures I-14 through I-17 and Table I-8.

CASE	INPUT DATA	
1	Initial A Group 1 Weapons	2
	Initial A Group 2 Weapons	10
	Initial B Group 1 Weapons	1
	Initial B Group 2 Weapons	5
	Rate A Group 1 Weapons Kill Acquired B Group 1 Weapons	.001
	Rate A Group 1 Weapons Kill Acquired B Group 2 Weapons	.002
	Rate A Group 2 Weapons Kill Acquired B Group 1 Weapons	.0002
	Rate A Group 2 Weapons Kill Acquired B Group 2 Weapons	.0004
	Rate B Group 1 Weapons Kill Acquired A Group 1 Weapons	.004
	Rate B Group 1 Weapons Kill Acquired A Group 2 Weapons	.008
	Rate B Group 2 Weapons Kill Acquired A Group 1 Weapons	.0008
	Rate B Group 2 Weapons Kill Acquired A Group 2 Weapons	.0016
	A Shift Coefficient	10.0
	B Shift Coefficient	12.0
	Rate an A Detects a Firing B	.03
	Rate a B Detects a Firing A	.05
	Rate an A Detects a Silent B	.015
	Rate a B Detects a Silent A	.025
	Rate an A Loses a Detected B	.01
	Rate a B Loses a Detected A	.01
	A Observer Autocorrelation	0.0
	B Observer Autocorrelation	0.0
2	Initial A's are Reduced by 50% in Each Group	
3	Each A Kill Rate is Reduced by 25%	
4	Rate A Detects a Firing B	.005
	Rate B Detects a Firing A	.005
	Rate A Detects a Silent B	.004
	Rate B Detects a Silent A	.004
5	Observer Autocorrelations	.25
6	Detection Rates Equal to Case 4, Observer Autocorrelations	.5
1H - 6H:	*Same as 1 - 6 with all group 1's eliminated.*	

Table I-7

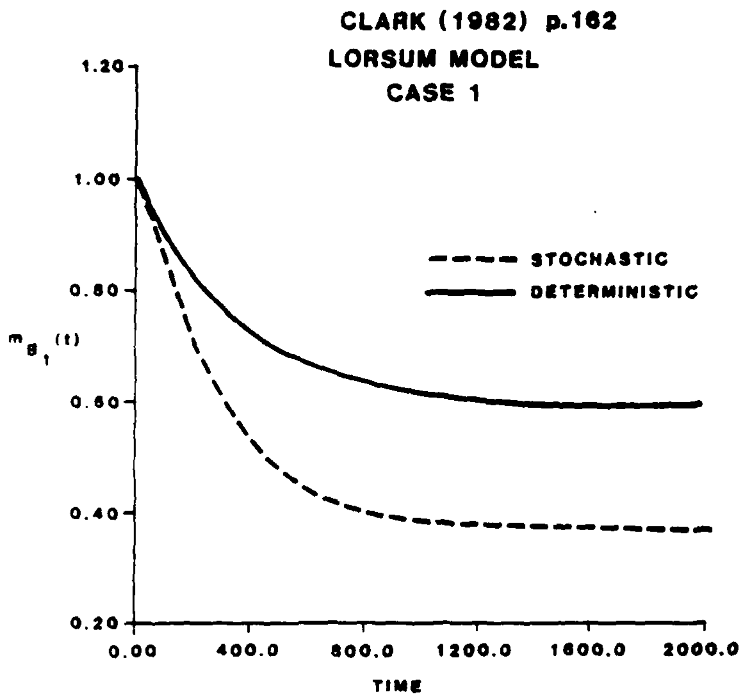


(a)

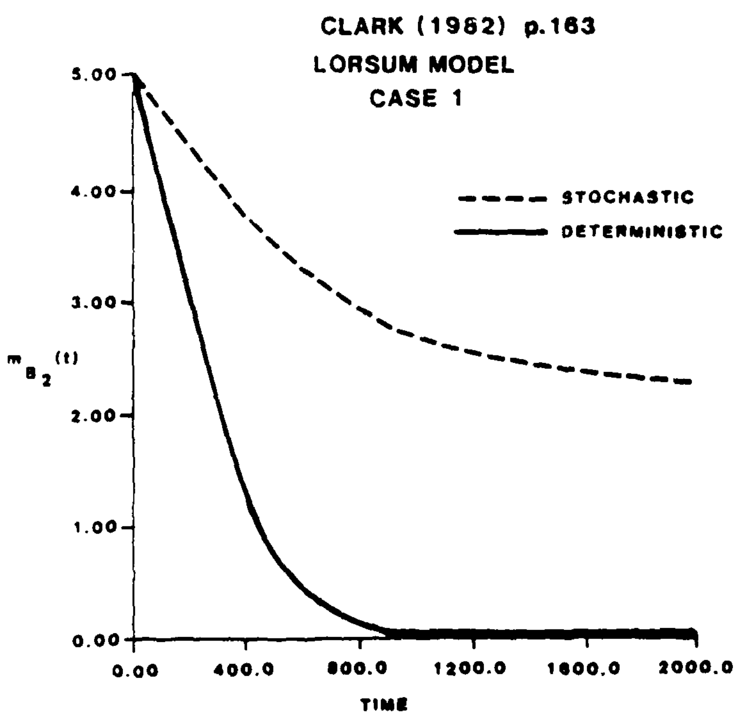


(b)

Figure I-14



(a)



(b)

Figure I-15.

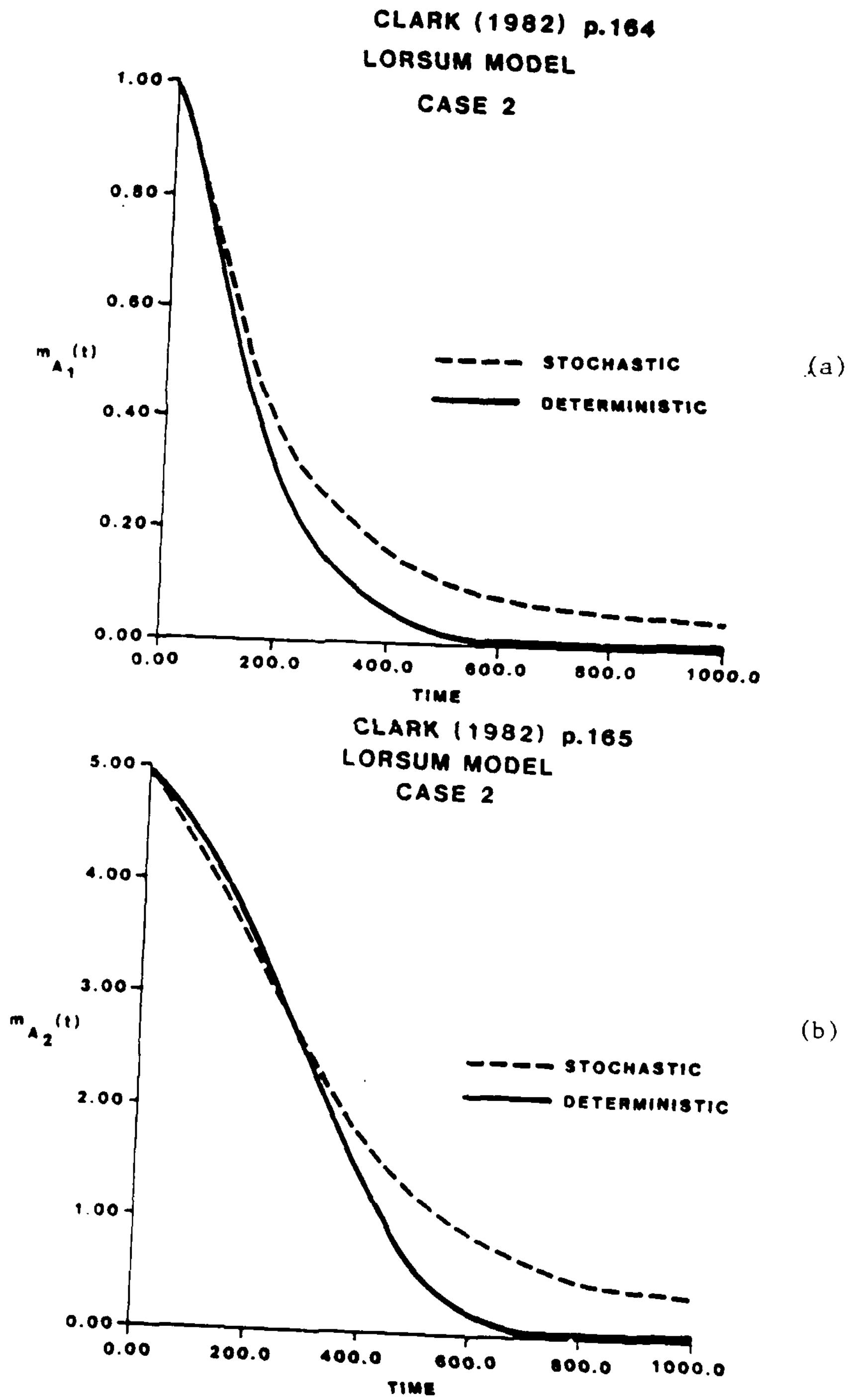
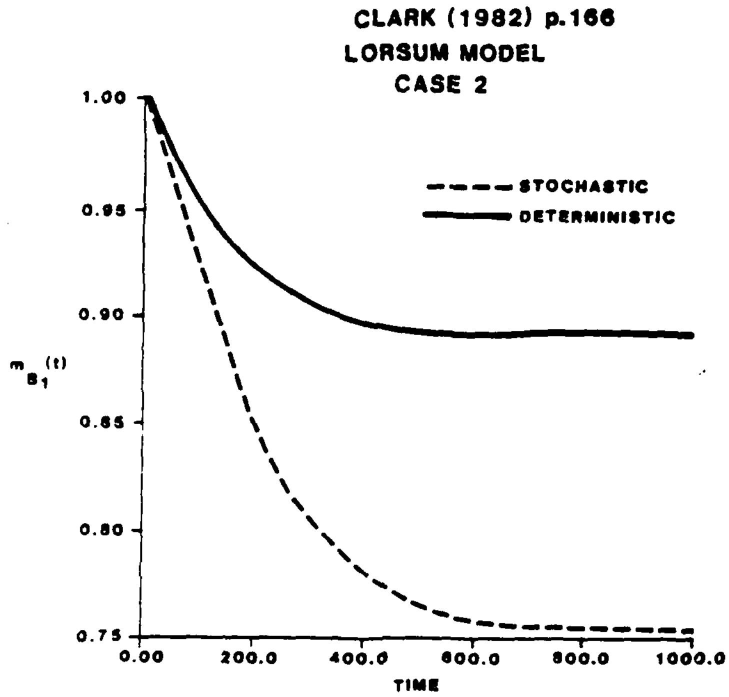
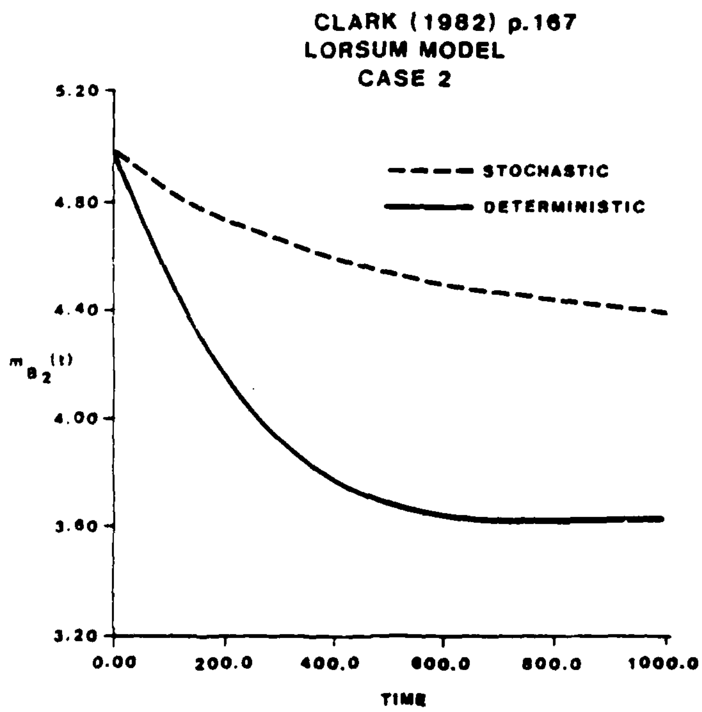

CLARK (1982) p.164
LORSUM MODEL
CASE 2
1.00
0.80
0.60
0.40
0.20
0.00
m_A1(t)
STOCHASTIC
DETERMINISTIC
(a)
0.00
200.0
400.0
600.0
800.0
1000.0
TIME
CLARK (1982) p.165
LORSUM MODEL
CASE 2
5.00
4.00
3.00
2.00
1.00
0.00
m_A2(t)
STOCHASTIC
DETERMINISTIC
(b)
0.00
200.0
400.0
600.0
800.0
1000.0
TIME

Figure I-16.



(a)



(b)

Figure I-17

CLARK (1982) pp.168-169

LORSUM MODEL

t = 500

CASE	A GROUP 1 SURVIVORS		B GROUP 1 SURVIVORS	
	STOCHASTIC	DETERMINISTIC ERROR	STOCHASTIC	DETERMINISTIC ERROR
1	.5046	-.0828	.4712	.2290
2	.1126	-.0879	.7659	.1298
3	.4311	-.2494	.5678	.2184
4	1.0179	.0351	.7018	.0977
5	.5274	-.2515	.4689	.2290
6	1.1318	-.0011	.6932	.1011

(a)

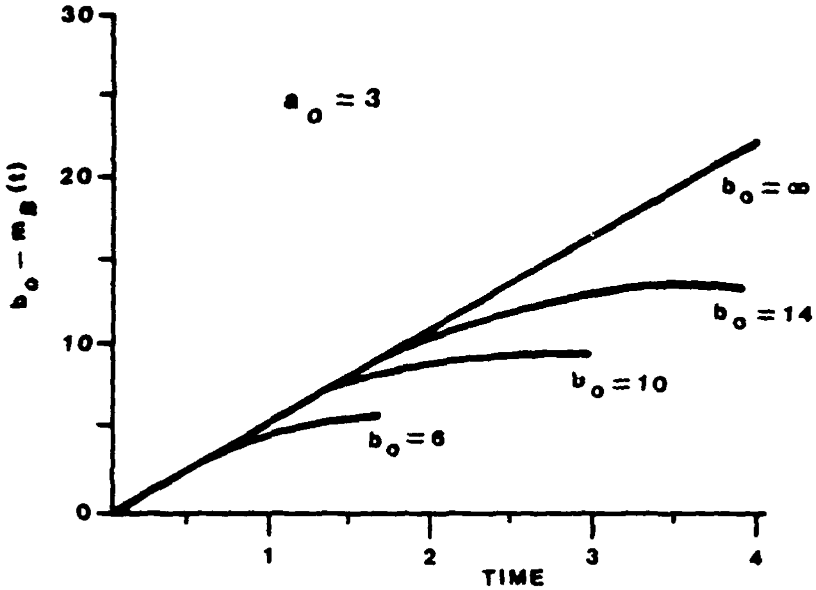
CASE	A GROUP 2 SURVIVORS		B GROUP 2 SURVIVORS	
	STOCHASTIC	DETERMINISTIC ERROR	STOCHASTIC	DETERMINISTIC ERROR
1	7.217	.602	3.535	-2.757
2	1.202	- .654	4.525	- .850
3	6.948	.626	3.979	-2.083
4	6.292	.425	3.463	-2.366
5	7.183	.814	3.571	-2.573
6	6.824	.247	3.787	-2.092
1H	6.807	2.152	3.492	-3.490
2H	1.757	1.084	4.379	-3.309
3H	6.535	2.080	3.827	-3.792
4H	6.960	1.507	3.837	-3.402
5H	6.694	2.216	3.489	-3.487
6H	7.499	1.014	4.103	-3.283

(b)

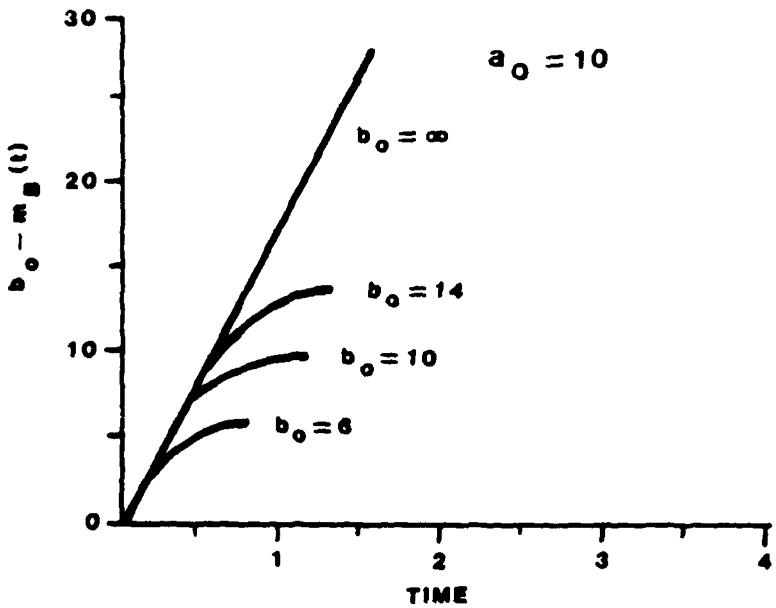
Table I-8

CHO (1984) SQUARE LAW (GR) p.48

a_o firers vs. b_o passive targets, uniform (o,1) interkill times
$b_o = \infty$, curves from theory, other curves from simulation
(40 replications)



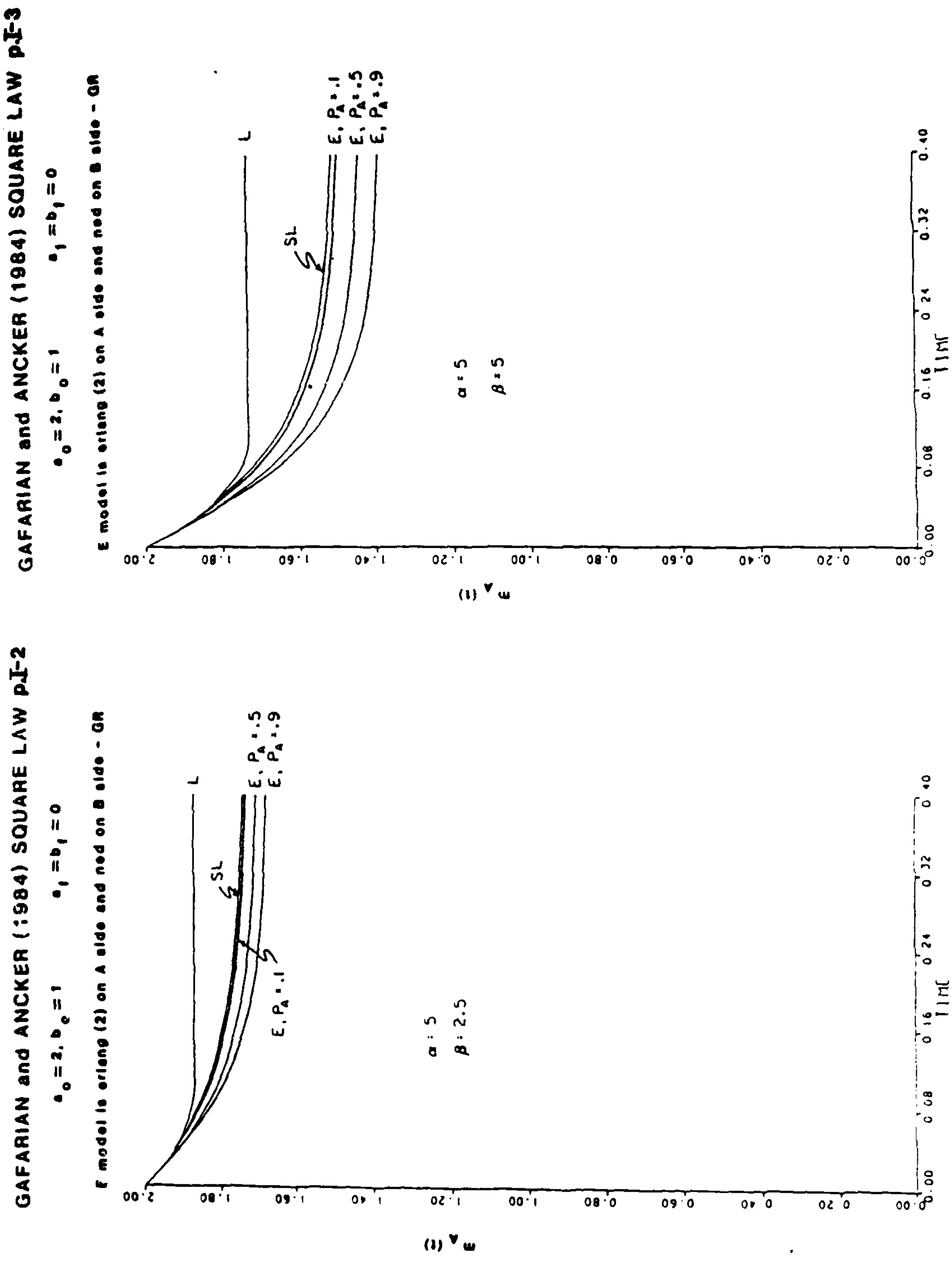
(a)



(b)

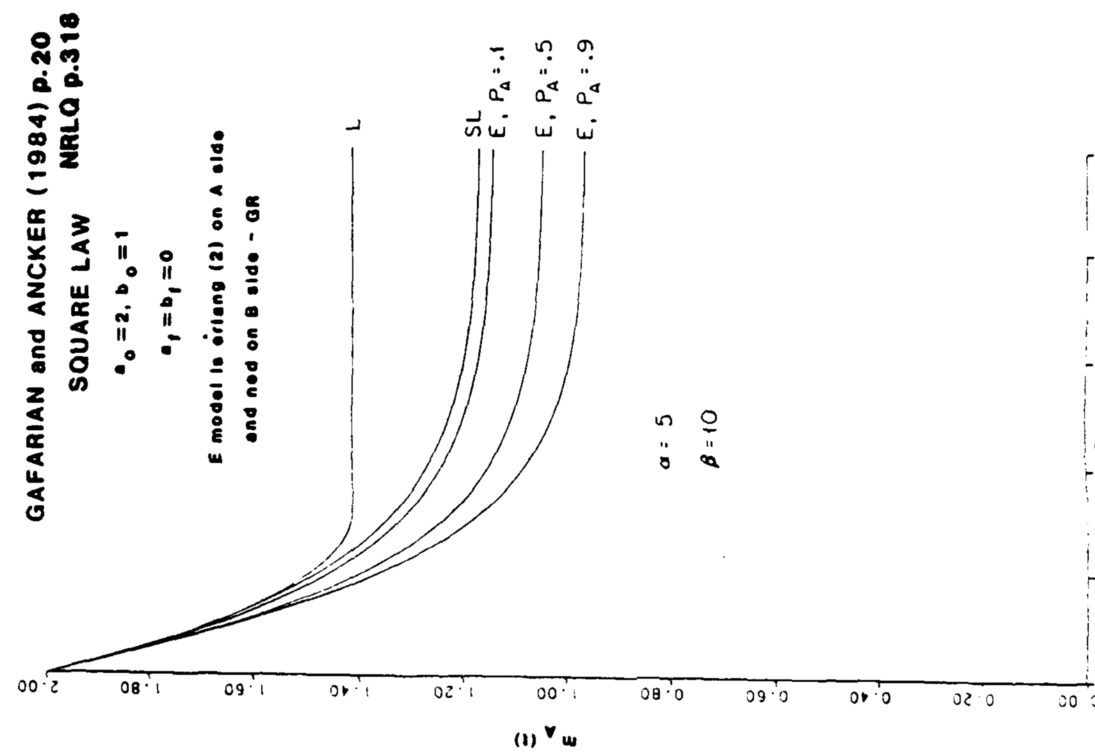
Figure I-18

(b)

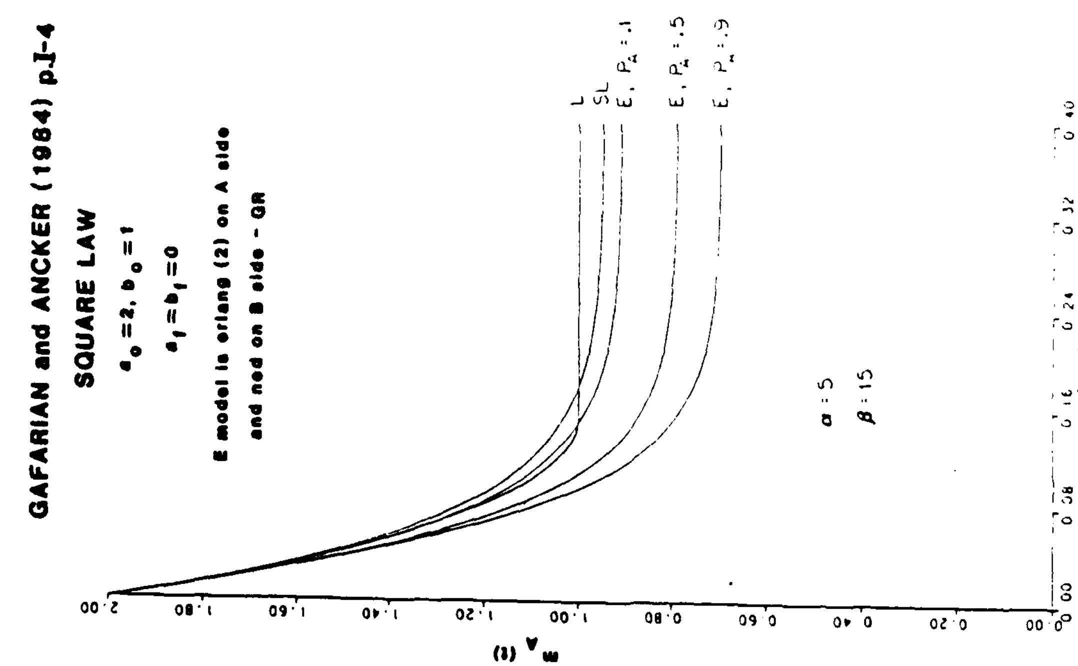


(a)

Figure I-19.



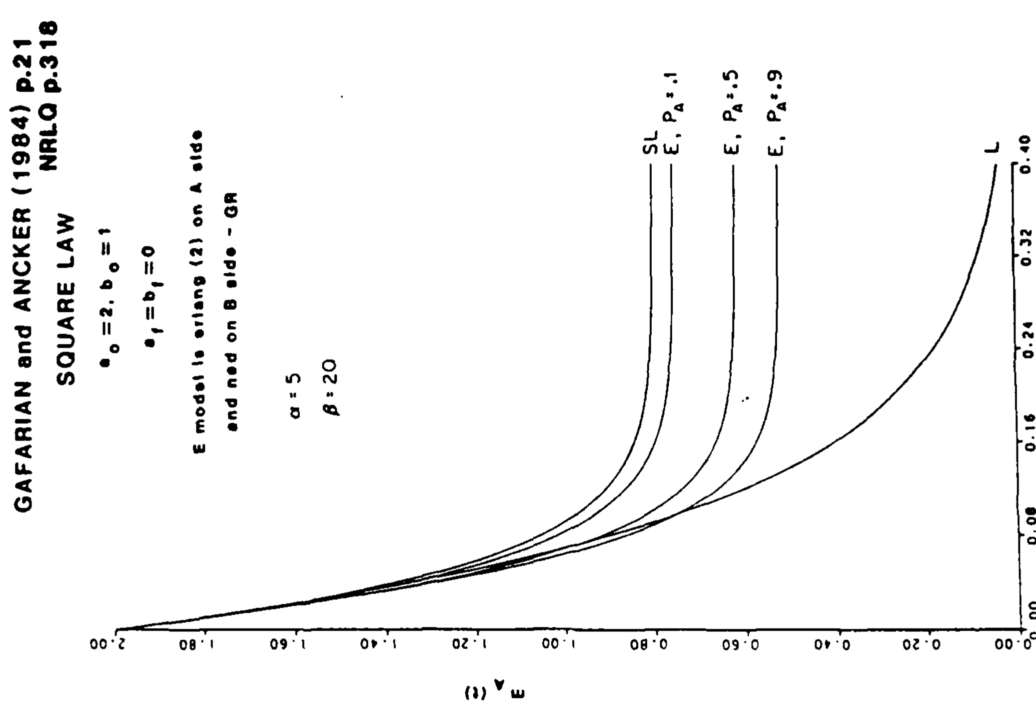
(b)

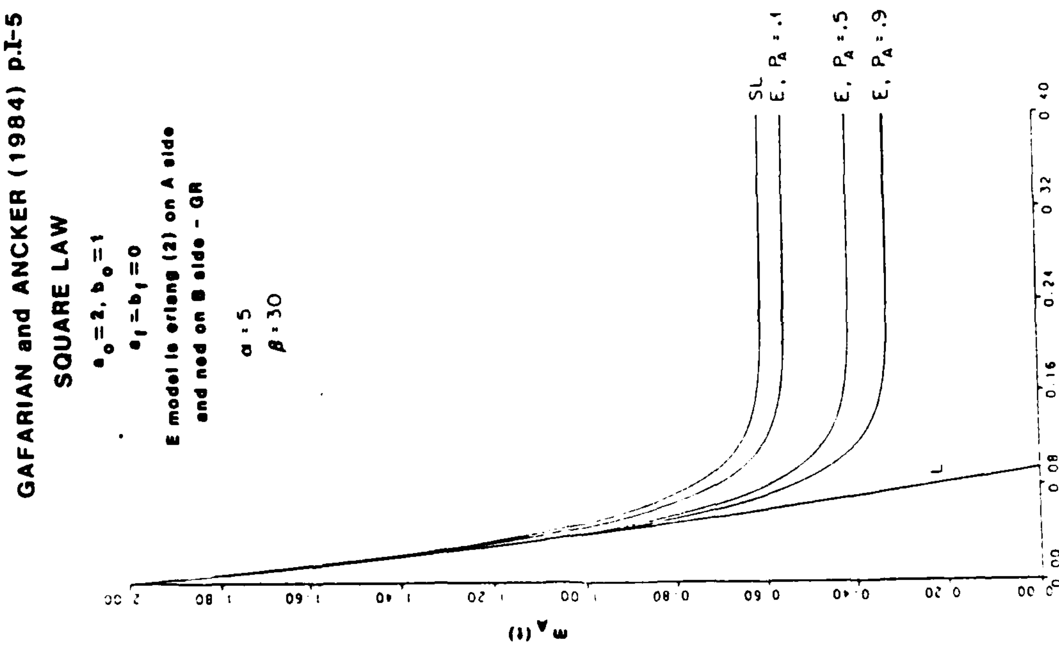


(a)

Figure I-20

(b)





(a)

Figure I-21.

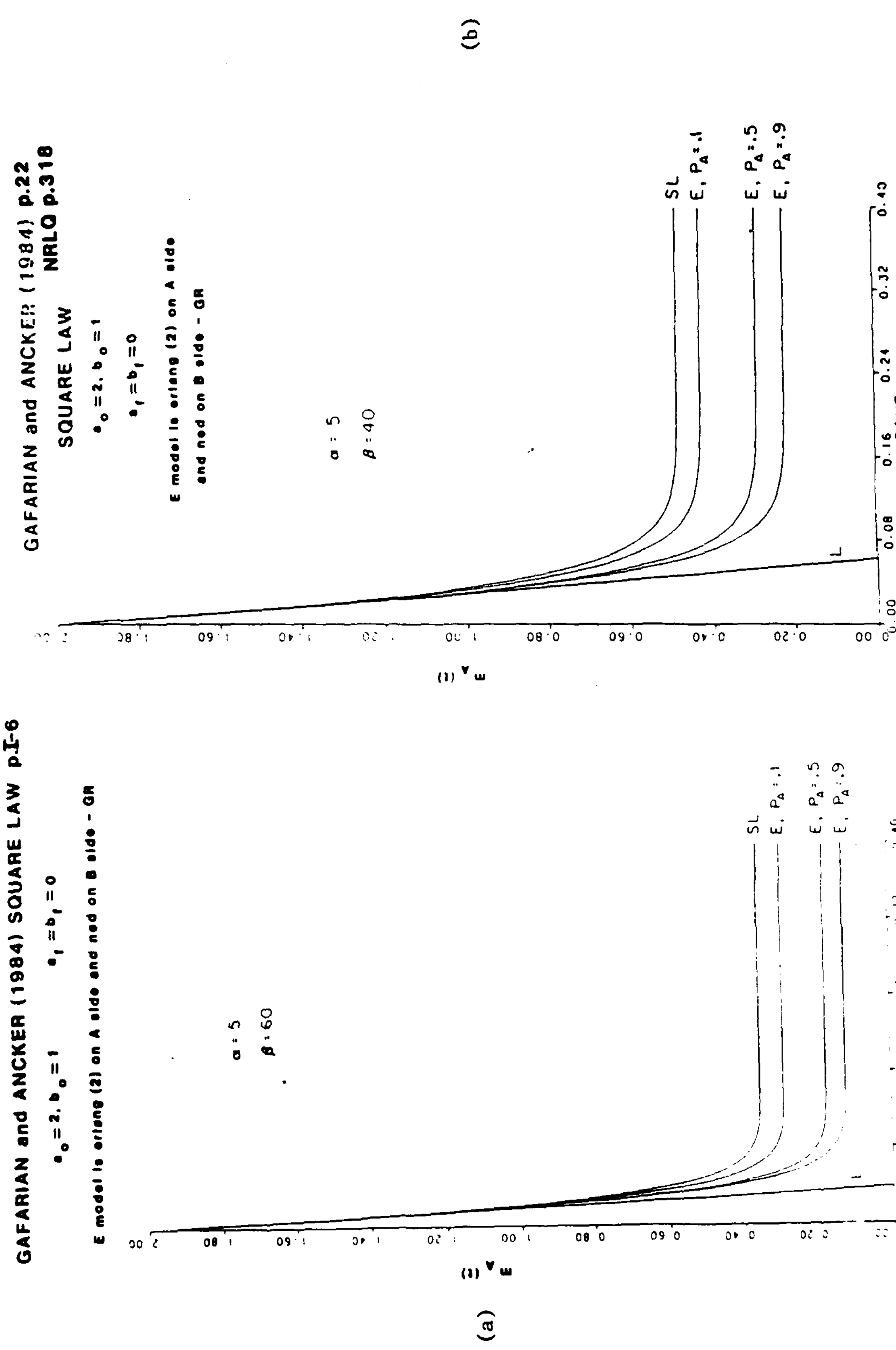

(b)
GAFARIAN and ANCKER (1984) p.22
SQUARE LAW
NRLQ p.318
a_o = 2, b_o = 1
a_1 = b_1 = 0
E model is erlang (2) on A side
and ned on B side - GR
α = 5
β = 40
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
L
TIME
m_A(t)
(a)
GAFARIAN and ANCKER (1984) SQUARE LAW p.I-6
a_o = 2, b_o = 1
a_1 = b_1 = 0
E model is erlang (2) on A side and ned on B side - GR
α = 5
β = 60
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
m_A(t)

Figure I-22.

GAFARIAN and ANCKER (1984) SQUARE LAW p.I-7

$a_o = 2, b_o = 1 \qquad a_1 = b_1 = 0$

E model is erlang (2) on A side and ned on B side - GR

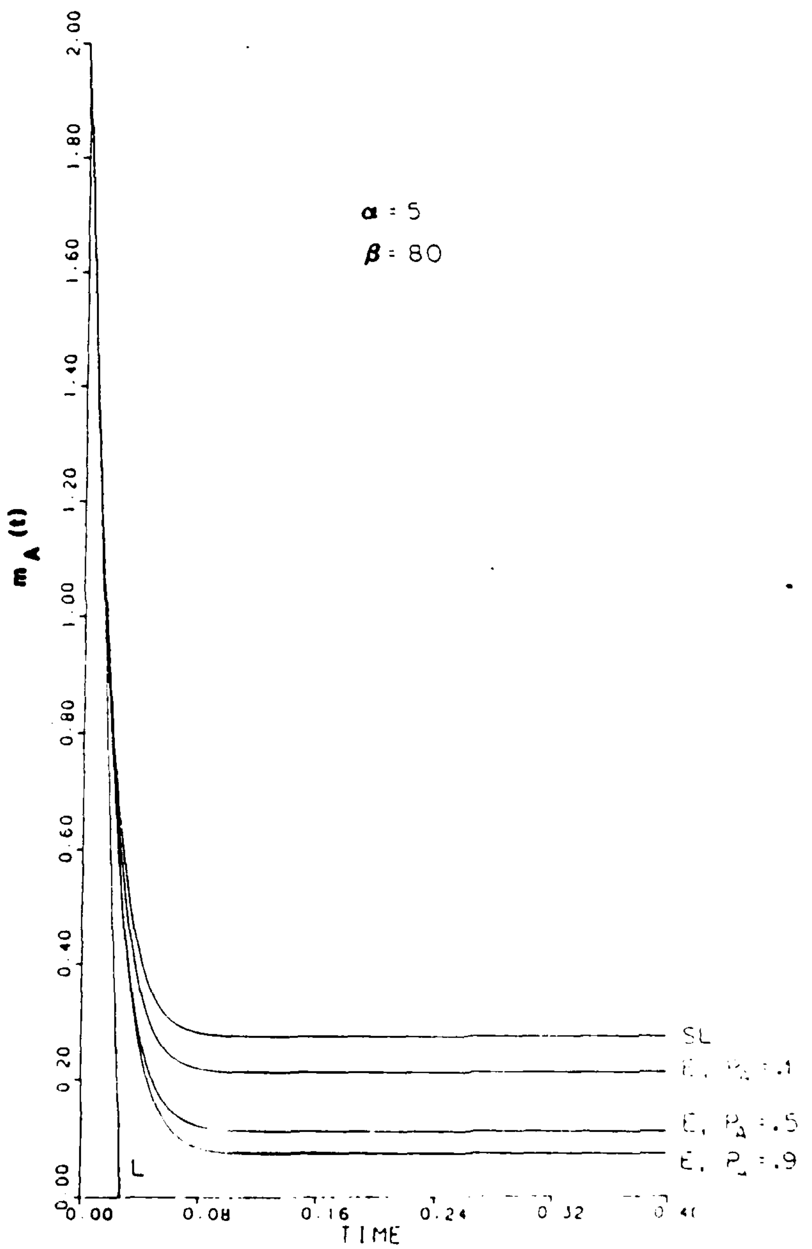


Figure I-23

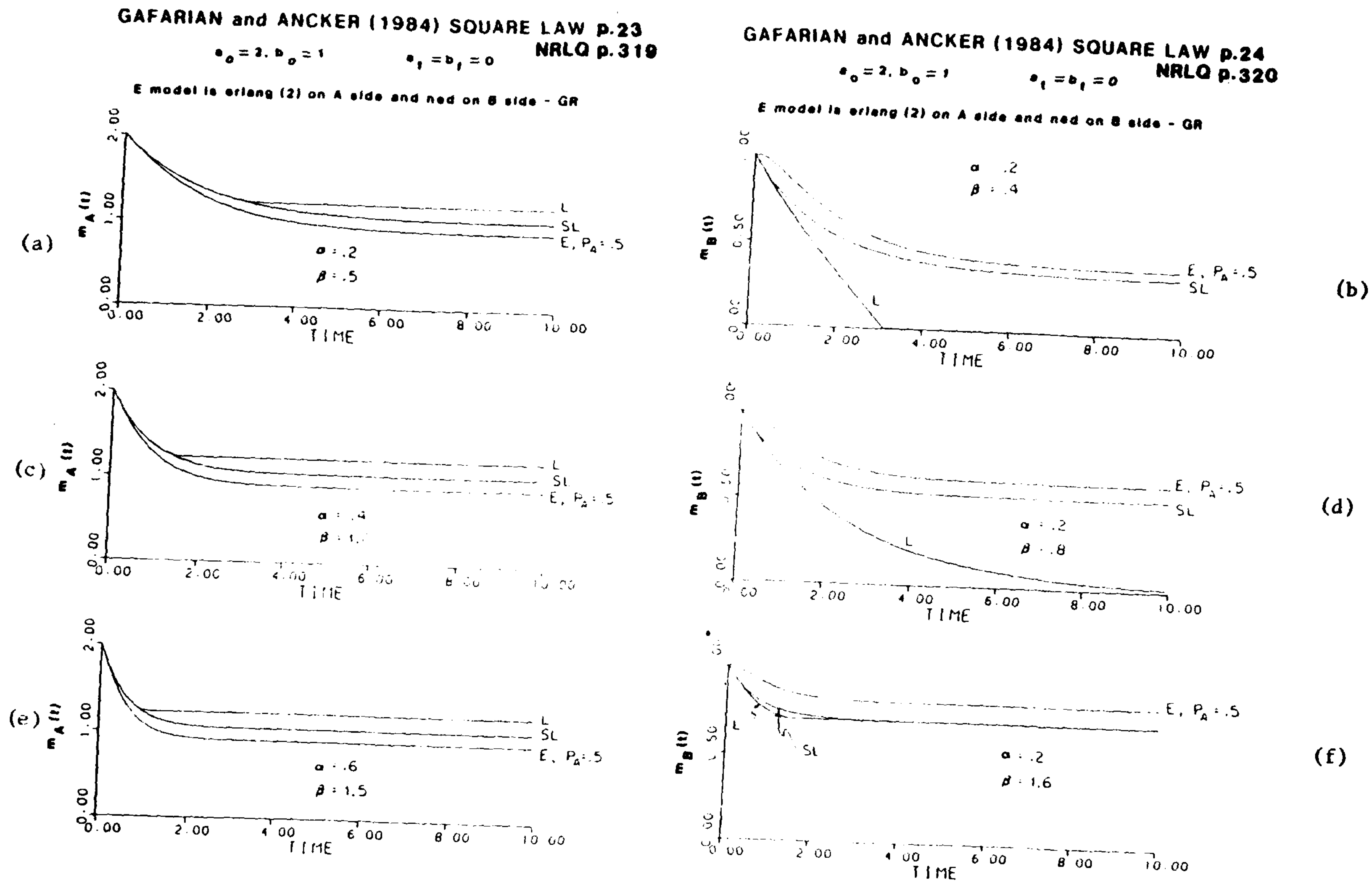

GAFARIAN and ANCKER (1984) SQUARE LAW p.23
NRLQ p.319
$a_o = 2, b_o = 1$
$a_1 = b_1 = 0$
E model is erlang (2) on A side and ned on B side - GR
(a)
$m_A(t)$
L
SL
E, P_A : .5
α : .2
β : .5
TIME
(c)
α : .4
(e)
α : .6
β : 1.5
GAFARIAN and ANCKER (1984) SQUARE LAW p.24
NRLQ p.320
(b)
$m_B(t)$
α : .2
β : .4
(d)
α : .2
β : .8
(f)
α : .2
β : 1.6
0.00
1.00
2.00
4.00
6.00
8.00
10.00

Figure I-24

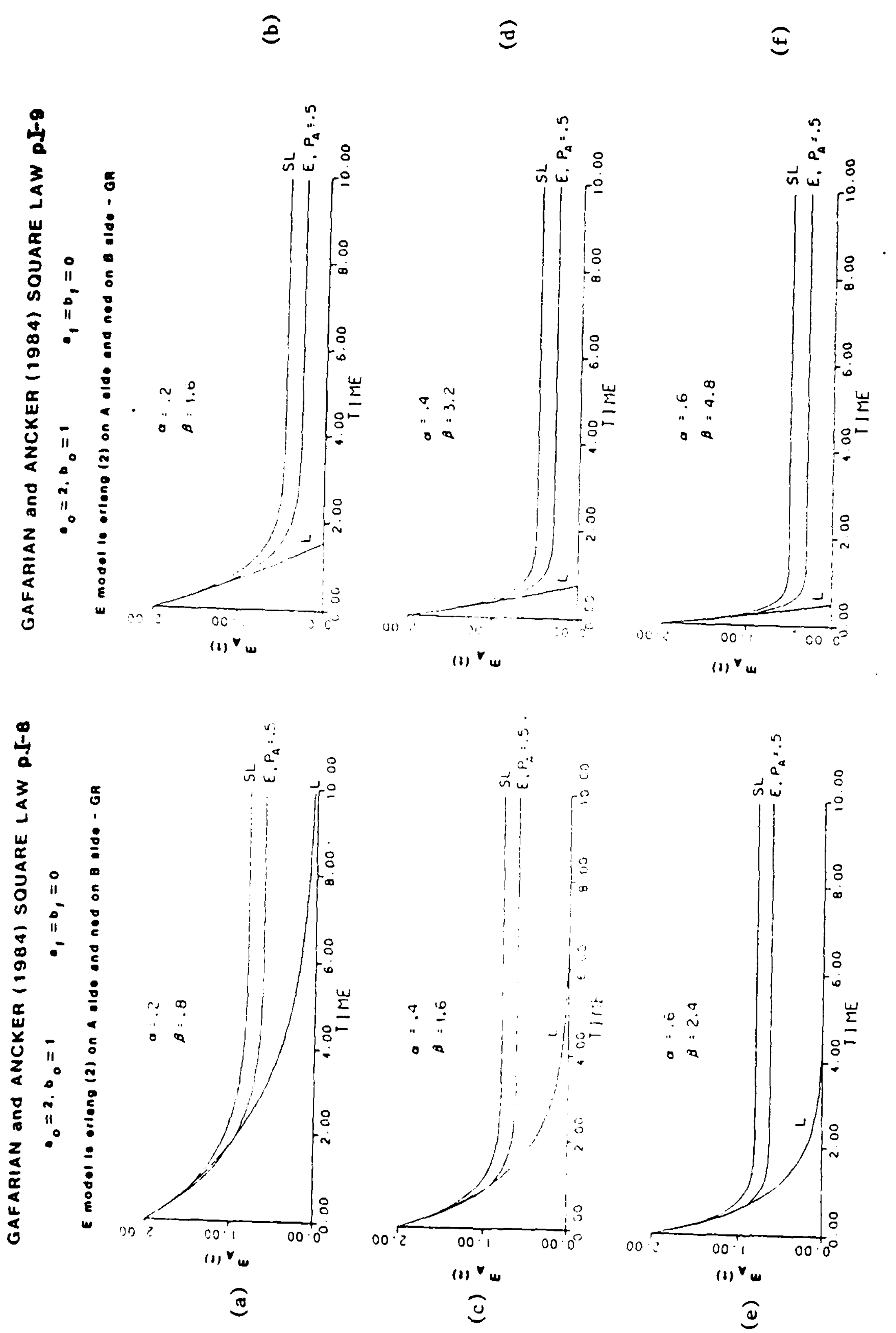

GAFARIAN and ANCKER (1984) SQUARE LAW p.I-8
$a_o = 2, b_o = 1$ $a_1 = b_1 = 0$
E model is erlang (2) on A side and ned on B side - GR
GAFARIAN and ANCKER (1984) SQUARE LAW p.I-9
$a_o = 2, b_o = 1$ $a_1 = b_1 = 0$
E model is erlang (2) on A side and ned on B side - GR
TIME
SL
E, P_A = .5
L
(a)
(b)
(c)
(d)
(e)
(f)

Figure I-25.

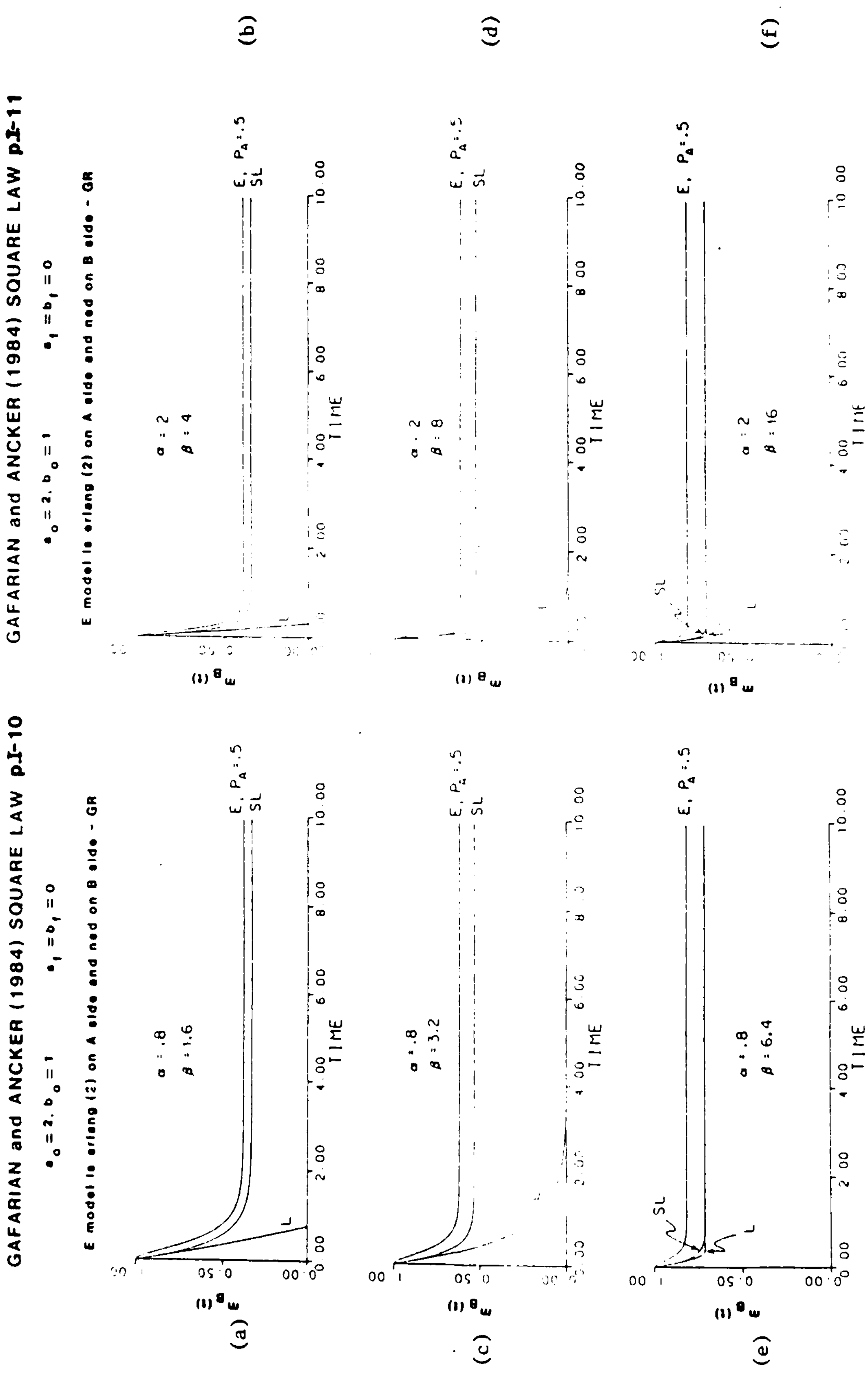

GAFARIAN and ANCKER (1984) SQUARE LAW pI-10
GAFARIAN and ANCKER (1984) SQUARE LAW pI-11
E model is erlang (2) on A side and ned on B side - GR
TIME
E, P_A: .5
SL
L
(a)
(b)
(c)
(d)
(e)
(f)

Figure I-26.

II. THE EXPECTED NUMBER OF TERMINAL SURVIVORS

This section is closely related to Section I and in fact is exactly the situation examined there at $t \to \infty$ only. It is singled out for particular attention because significant discrepancies between terminal L and SL (or GR) mean values should be especially important to decision-makers and other users.

Most of the figures and tables in Section I and Figures VI-3 through VI-12 have been carried far enough in time so that terminal differences are very closely approximated. This is true except for the following:

(1) Where there is no deteministic information given; i.e. Figure I-8 and Tables I-2, I-3(a) and I-6.

(2) Where time is not carried far enough; i.e. Figures I-9 and I-12 and Table I-3(b).

In addition to the parity cases noted in Section I we also have parity here in Figures II-2 and II-3. Again we note that $x(\infty) = y(\infty) = 0$.

The inadequacies of L and SL models as previously noted on pages 15 and 16 and in Part Two, Section I are even more apparent here.

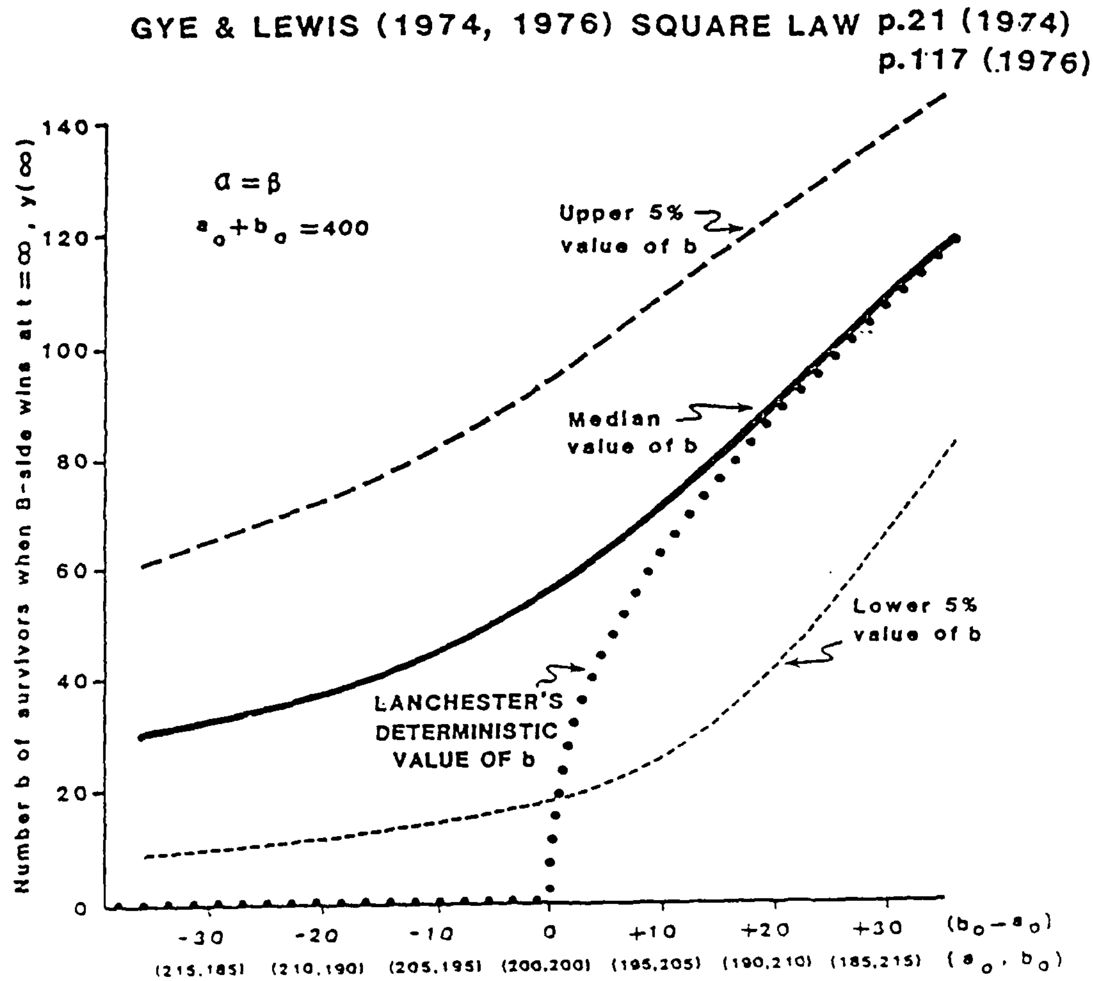
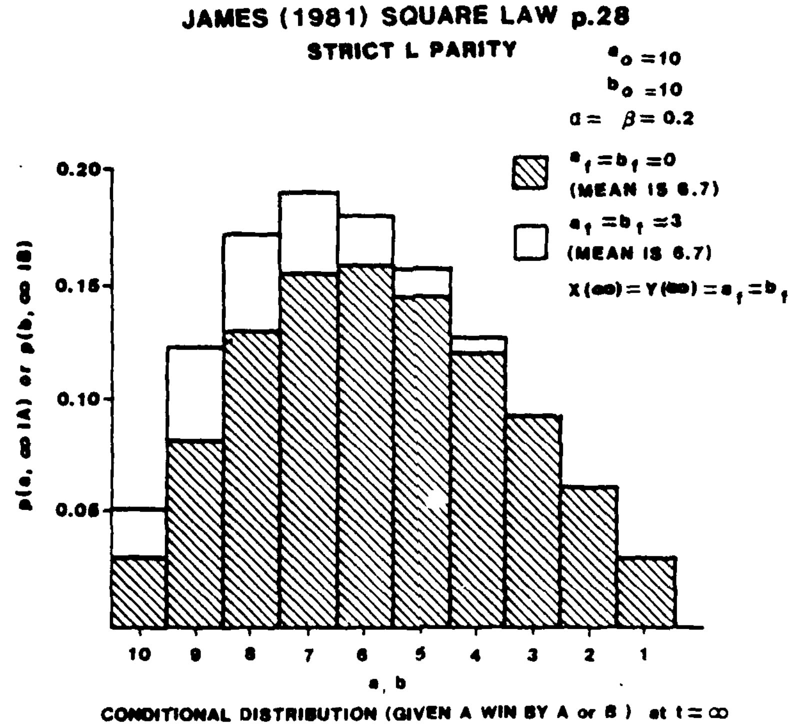
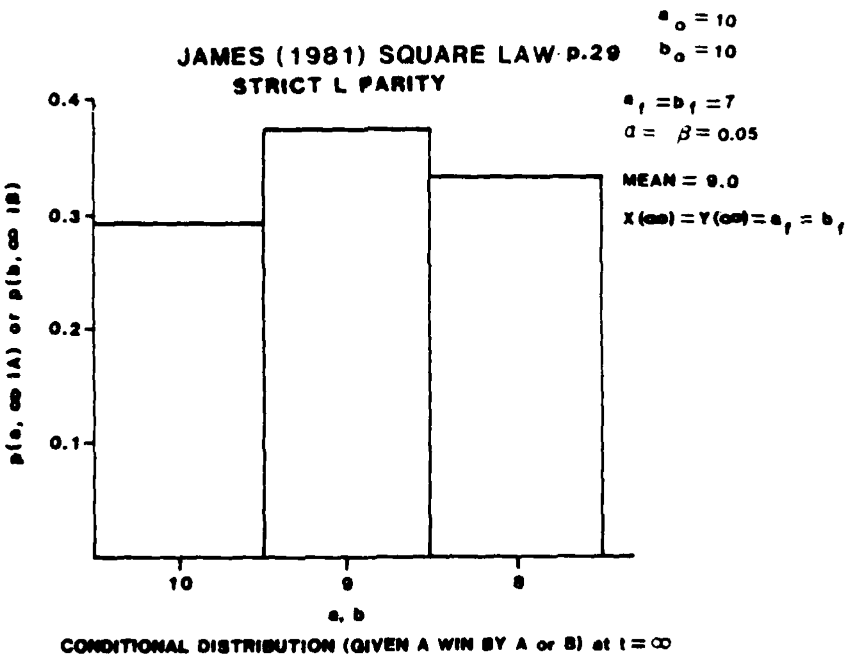

GYE & LEWIS (1974, 1976) SQUARE LAW p.21 (1974)
p.117 (1976)
α = β
$a_o + b_o = 400$
Upper 5% value of b
Median value of b
Lower 5% value of b
LANCHESTER'S DETERMINISTIC VALUE OF b
Number b of survivors when B-side wins at t = ∞, y(∞)
140
120
100
80
60
40
20
0
-30
-20
-10
0
+10
+20
+30
$(b_o - a_o)$
(215,185)
(210,190)
(205,195)
(200,200)
(195,205)
(190,210)
(185,215)
(a_o, b_o)

Figure II-1



(a)



(b)

Figure II-2

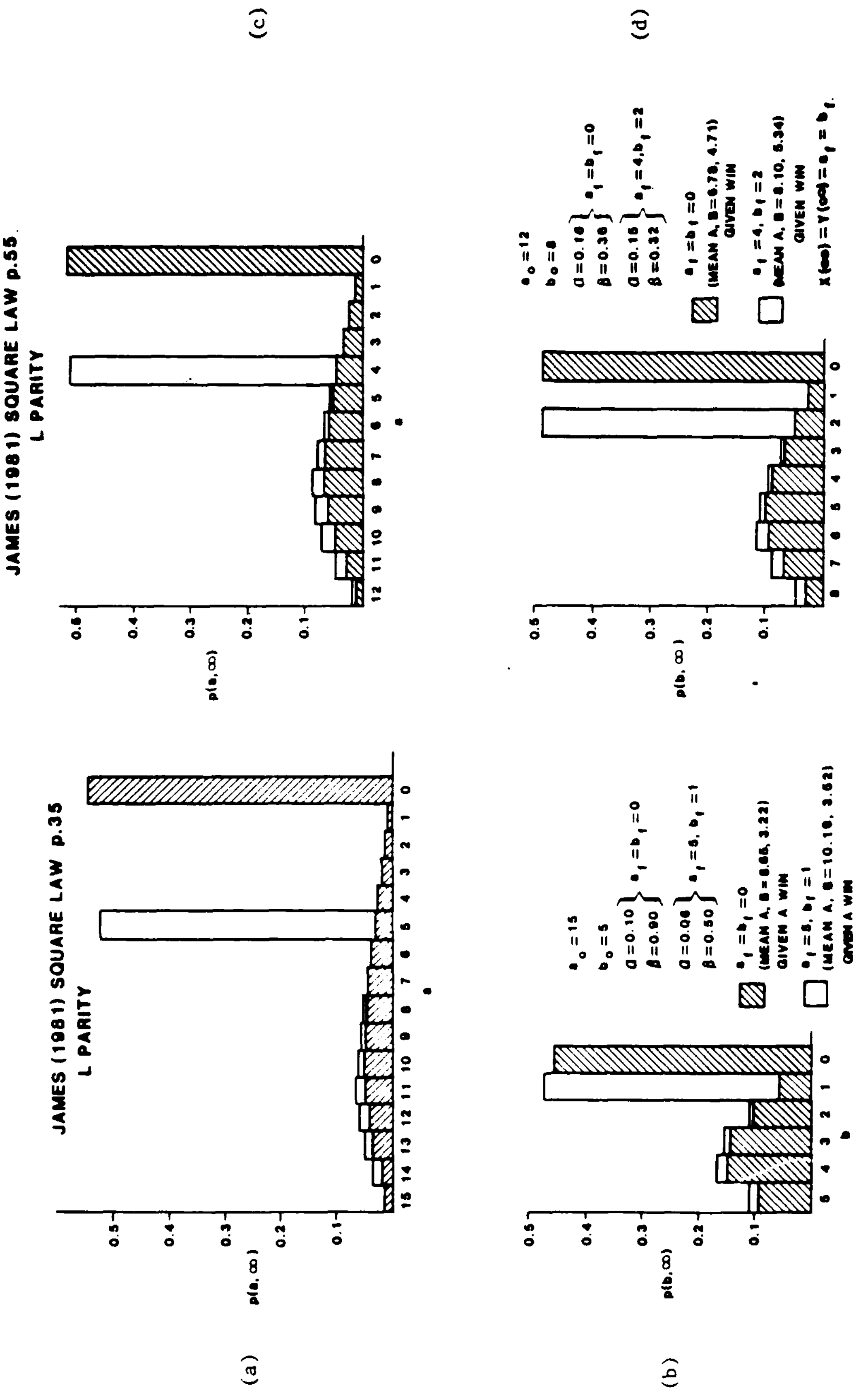

JAMES (1981) SQUARE LAW p.35
L PARITY
JAMES (1981) SQUARE LAW p.55
L PARITY
(a)
(b)
(c)
(d)
Marginal Distributions at t = ∞

Figure II-3

MORSE AND KIMBALL (1956) p.70

SQUARE LAW

$a_0 = 5,\ b_0 = 3$

$a_f = b_f = 0$

$\alpha = \beta$

p(a,b,∞)	
$P(5,0,\infty) = .3721$	$P(0,3,\infty) = .0362$
$P(4,0,\infty) = .2690$	$P(0,2,\infty) = .0469$
$P(3,0,\infty) = .1456$	$P(0,1,\infty) = \underline{.0295}$
	$P(B) = .1126$
$P(2,0,\infty) = .0712$	
$P(1,0,\infty) = \underline{.0295}$	$y(\infty) = 0$
$P(A) = .8874$	$E[B(\infty)] = 0.2312$
$x(\infty) = 1$	
$E[A(\infty)] = 3.4913$	

(a)

WALLIS (1967) p.15

SQUARE LAW

$a_0 = 3,\ b_0 = 2$

$a_f = b_f = 0$

$\alpha = \beta$

a	b	P(a,b,)
3	0	= 54/120
2	0	= 29/120
1	0	= 11/120
0	1	= 11/120
0	2	= 16/120
$m_A(\infty) = 1.91$		
$x(\infty) = 2.28$		
$m_B(\infty) = .36$		
$y(\infty) = 0$		
$\Delta_A(\infty) = -.37$		
$\Delta_B(\infty) = .36$		
$P(A) = .775$		
$P(B) = .225$		

(b)

Table II-1

JAMES (1981) p.37

SQUARE LAW

a_0	b_0	a_f	b_f	α	β	E[A(∞)|A WINS]	L P(A)	SL P(A)	E[A(∞)] = $m_A(\infty)$	x(∞)	E[B(∞)|B WINS]	L P(B)	SL P(B)	E[B(∞)] = $m_B(\infty)$	y(∞)	L P(B)
10	10	0	0	0.20	0.20	5.72	0	0.500	2.86	0.00	5.72	0	0.500	2.86	0.00	1
		3	3			6.70	0	0.500	4.85	3.00	6.70	0	0.500	4.85	3.00	1
		7	7	0.05	0.05	8.96	0	0.500	7.98	7.00	8.96	0	0.500	7.98	7.00	1
12	8	0	0	0.16	0.36	6.78	0	0.485	3.29	0.00	4.72	0	0.515	2.42	0.00	1
		4	2	0.15	0.32	8.10	0	0.488	6.00	4.00	5.34	0	0.512	3.71	2.00	1
15	5	0	0	0.10	0.90	8.65	0	0.459	3.97	0.00	3.22	0	0.541	1.74	0.00	1
		5	1	0.06	0.50	10.15	0	0.472	7.45	5.00	3.52	0	0.528	2.33	1.00	1
30	30	0	0	0.20	0.20	13.24	0	0.500	6.62	0.00	13.24	0	0.500	6.62	0.00	1
		9	9			16.47	0	0.500	12.74	9.00	16.47	0	0.500	12.74	9.00	1
		21	21	0.05	0.05	24.65	0	0.500	22.83	21.00	24.66	0	0.500	22.33	21.00	1
45 ·	15	0	0	0.10	0.90	20.50	0	0.475	9.74	0.00	7.34	0	0.525	3.85	0.00	1
		15	3	0.06	0.90	25.73	0	0.482	20.17	15.00	6.21	0	0.518	3.70	3.00	1
10	10	0	0	0.20	0.20	6.47	1	0.787	5.09	5.77	5.17	0	0.213	1.10	0.00	0
		3	3			7.22	1	0.763	6.22	6.27	6.11	0	0.237	1.78	3.00	0
		7	7	0.075	0.05	9.80	1	0.682	8.42	8.12	8.06	0	0.318	7.50	7.00	0
15	5	0	0	0.15	0.90	9.61	1	0.728	6.99	8.66	2.99	0	0.272	0.81	0.00	0
		5	1	9.00	0.50 ·	10.85	1	0.720	9.21	9.57	3.33	0	0.280	1.65	1.00	0
30	30	0	0	0.30	0.30	17.41	1	0.914	15.91	17.32	10.74	0	0.086	0.93	0.00	0
		9	9			19.43	1	0.894	18.33	18.81	14.72	0	0.106	9.61	9.00	0
		21	21	0.075	0.50	25.52	1	0.800	24.61	24.37	24.07	0	0.200	21.61	21.00	0
45	15	0	0	0.15	0.90	25.96	1	0.875	22.71	25.98	6.17	0	0.125	0.77	0.00	0
		15	3	0.90	0.50	29.53	1	0.863	27.54	28.72	7.25	0	0.137	3.50	3.00	0
20	5	0	0	0.10	0.90	13.25	1	0.825	10.93	13.23	2.00	0	0.175	0.50	0.00	0
		6	1	0.00	0.50	14.56	1	0.829	13.11	14.14	3.22	0	0.171	1.34	1.00	0
60	15	0	0	0.10	0.90	38.09	1	0.959	36.52	39.69	5.75	0	0.041	0.24	0.00	0
		16	3	0.40	0.50	41.63	1	0.959	40.66	42.43	6.82	0	0.041	3.16	3.00	0

(a)

SPRINGALL (1968) p.49

SQUARE LAW

$a_0 = 10,\ b_0 = 8,\ \alpha = 1,\ \beta = 1$

$a_f = b_f = 0$

N.B. $x(\infty) = 6.0,\ y(\infty) = 0$

a OR b	1	2	3	4	5	6	7	8	9	10	TOTAL P(A), P(B)	$\frac{m_A(\infty)}{m_B(\infty)}$
p(a,a,∞)	0.01299	0.02696	0.04335	0.06327	0.08696	0.11254	0.13404	0.13972	0.11486	0.05668	0.79136	5.217
p(b,b,∞)	0.01299	0.02498	0.03442	0.03962	0.03892	0.03163	0.01932	0.00676			0.20864	.898

(b)

KISI (1965) p.52

SQUARE LAW

$a_0 = 3,\ b_0 = 2$

$a_f = b_f = 0$

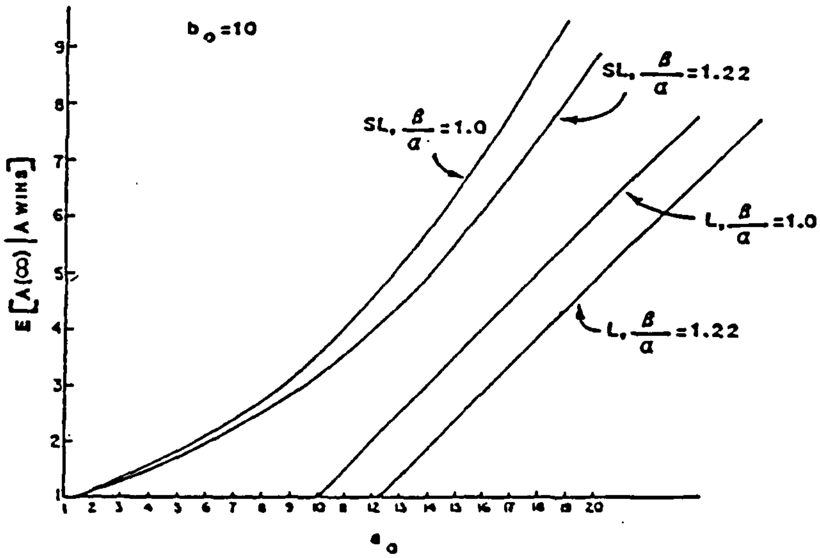
$\alpha = \beta$

LAW	p(a,a,∞) 3	2	1	P(A)	E[A(∞)|A]
SL Square	$\frac{9}{20}$	$\frac{7}{30}$	$\frac{11}{120}$	0.775	2.46
L Square				1.0	2.236 ·

(c)

Table II-2

WEISS (1963) LINEAR LAW p.601



CONDITIONAL MEANS

Figure II-4

KISI (1965) p.52

LINEAR LAW

$a_0 = 3,\ b_0 = 2$

$a_f = b_f = 0$

$\alpha = \beta$

LAW	$p(a,0,\infty)$ 3	2	1	P(A)	$E[A(\infty)\|A]$
SL Linear	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{16}$	0.688	2.09
L Linear				1.0	1.0

Table II-3

SPRINGALL (1968) p.158

SPRINGALL MODEL

$m_z = 5,\ \alpha = 0.9,\ \beta = 3.0,\ \delta = 2.0,\ \gamma = 2.0$

$a_f = b_f = 0$

$t = \infty$

$b_0 = a_0$	$P(B)$	EXPECTED NUMBER OF B SURVIVORS, b		EXPECTED NUMBER OF A SURVIVORS, a	
		DETERMINISTIC	STOCHASTIC	DETERMINISTIC	STOCHASTIC
5	0.88567	2.500	2.842	0	0.203
10	0.97080	5.587	5.854	0	0.059
20	0.99763	11.762	12.003	0	0.005
30	0.99979	17.937	18.117	0	0.001
40	0.99998	24.112	24.353*	0	0.000
50	1.00000	30.287	30.529	0	0.000
60	1.00000	36.462	36.706	0	0.000

(a)

SPRINGALL (1968) p.159

SPRINGALL MODEL

$a_0 = 50,\ b_0 = 40,\ m_z = 50,\ \alpha = \beta = 1,\ \gamma = \delta = 10$

$a_f = b_f$	$P(B)$	EXPECTED NUMBER OF SURVIVORS $m_B(\infty)$	
		DETERMINISTIC	STOCHASTIC
0	0.14455	0	0.704
1	0.14180	1	1.680
2	0.13900	2	2.656
4	0.13321	4	4.608
6	0.12715	6	6.560
8	0.12082	8	8.512
10	0.11420	10	10.465
20	0.07620	20	20.243
30	0.03071	30	30.066

(b)

SPRINGALL (1968) p.177

SPRINGALL MODEL

$a_0 = 60,\ b_0 = 60,\ \alpha = 0.9,\ \beta = 1.0,\ \gamma = 2.0,\ \delta = 2.0$

$a_f = b_f = 5$

m_z and n_z	$m_A(\infty)$	$m_B(\infty)$	$x(\infty)$	$y(\infty)$
10	7.31188	11.68936	0	9.64
15	7.22422	11.84205	0	9.85
20	7.18098	11.91737	0	9.95
30	7.14259	11.98471	0	10.04
40	7.12821	12.01016	0	10.07
50	7.12287	12.01975	0	10.08
60	7.12149	12.02212	0	10.08

(c)

Table II-4

III. THE EXPECTED TIME-DURATION OF THE COMBAT

The following do not give L termination times for comparison with $E[T_D]$ but are included for completeness; Tables III-2 and III-3.

Certain figures in Section I contain information on time-duration and should be consulted in connection with the material given here. They are Figures I-1(b), (c) (these are parity cases where $t_f \to \infty$), I-2, I-6, I-7, and I-8 which are all Square Law.

The following figures clearly indicate the effect of breakpoints when compared to annihilation models: Figures III-1(b), III-2(b), III-3(a), III-4 and Table III-1(b). The breakpoints lower the time-duration of the fire-fight substantially as one would expect.

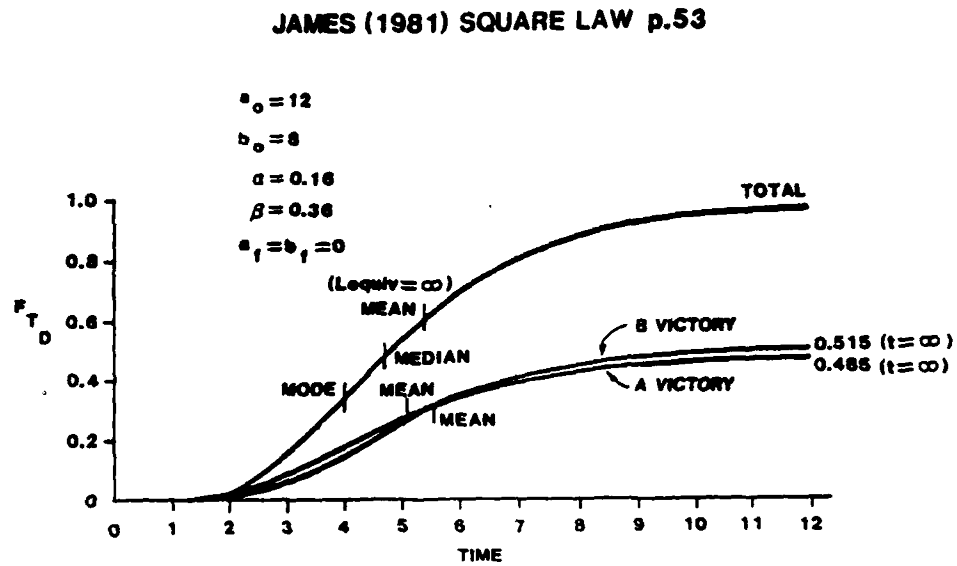
Nowhere is the variance of T_D explicitly computed except in Tables III-1(b), (c). The 5th and 95th percentiles are shown in Table III-1(b). However, these tables and an inspection of all the probability distribution functions (fairly flat) and density functions (rather spread out) demonstrate that $V[T_D]$ is indeed substantial. This fact and the large discrepancies between $E[T_D]$ and t_f which can be extremely large near parity (where $t_f \to \infty$) lead to the conclusion that t_f is an extremely poor approximation to the measure, expected time-duration.

Finally, Figures III-1 and III-2 not only show plots of the distribution function of T_D but also the marginal functions $P[T_D < t, \text{A wins}]$ and $P[T_D < t, \text{B wins}]$. We point out that these latter two are not conditionals (some of which are shown in Tables III-2 and III-3(a)), but are improper distribution functions with the property that as $t \to \infty$ these function tend to $\sum_{a=a_f}^{a_0} p(a,0,\infty) = P(A)$ and $\sum_{b=b_f}^{b_0} p(0,b,\infty) = P(B)$. This is related to the fact that the expression at the bottom of page 36 is the time derivative of the identity;

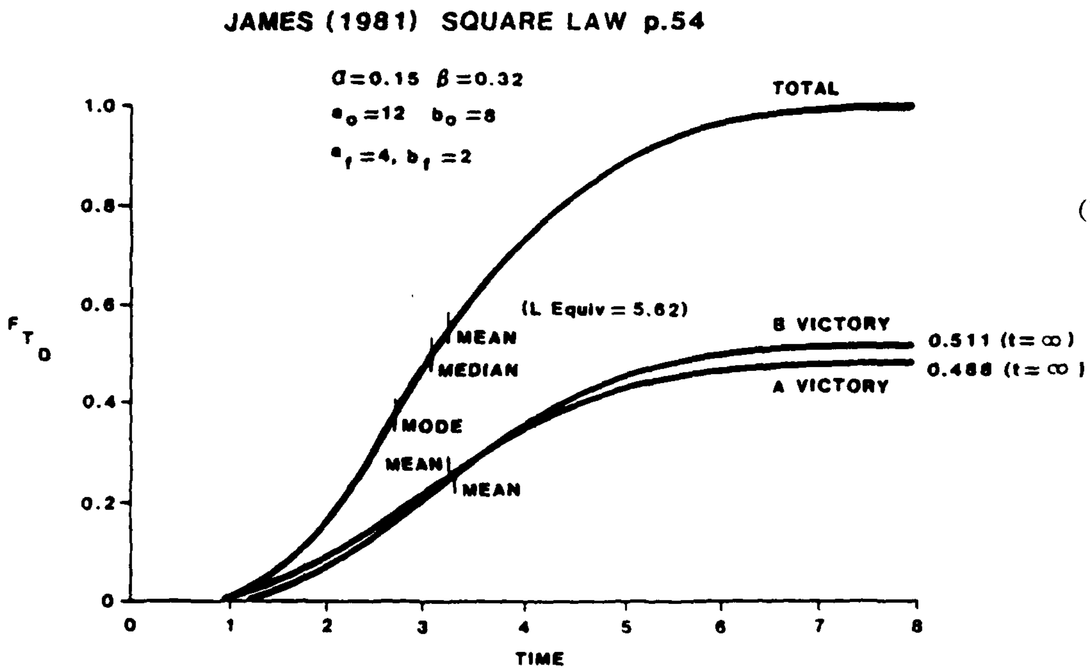
$$F_{T_D}(t) = P[T_D < t] = P[T_D < t, \text{A wins}] + P[T_D < t, \text{B wins}]$$

$$= \sum_{a=a_f}^{a_0} p(a,0,t) + \sum_{b=b_f}^{b_0} p(0,b,t)$$

which, of course, only involves the absorption probabilities.

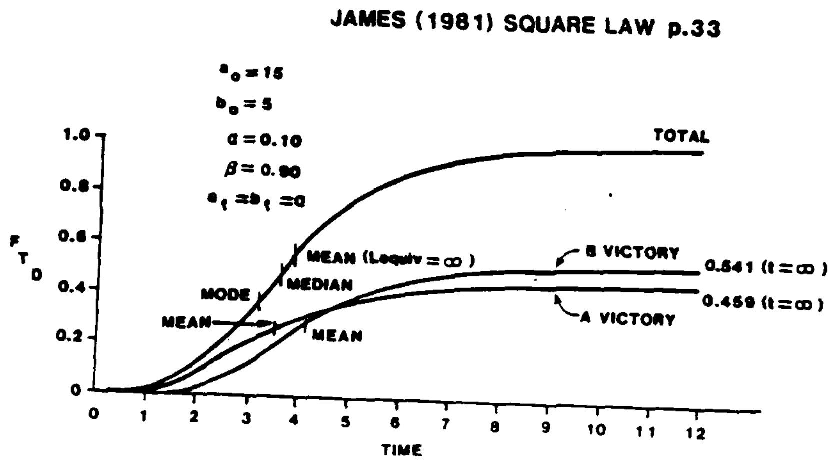


(a)

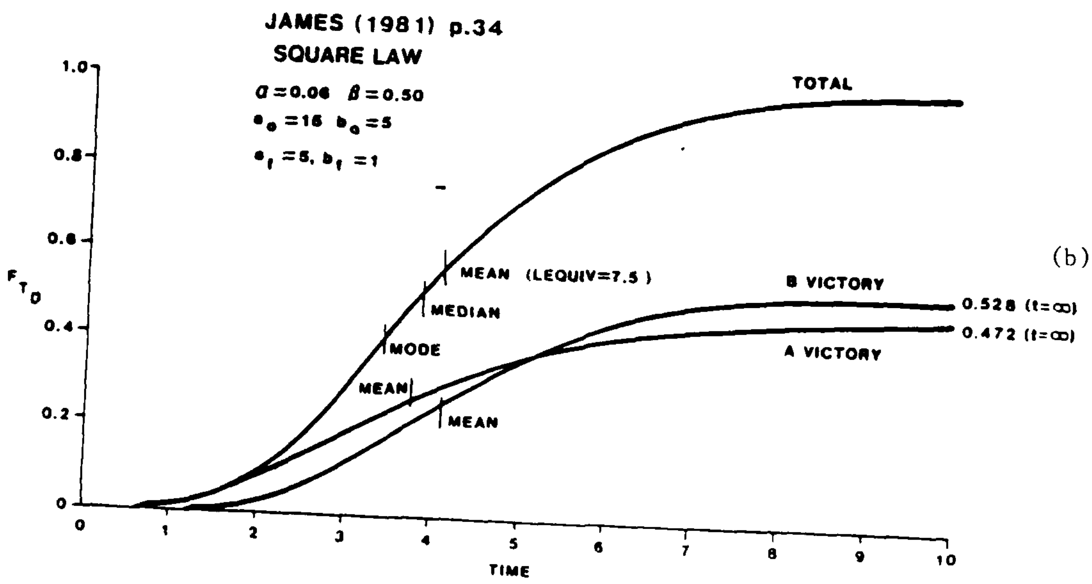


(b)

Figure III-1.

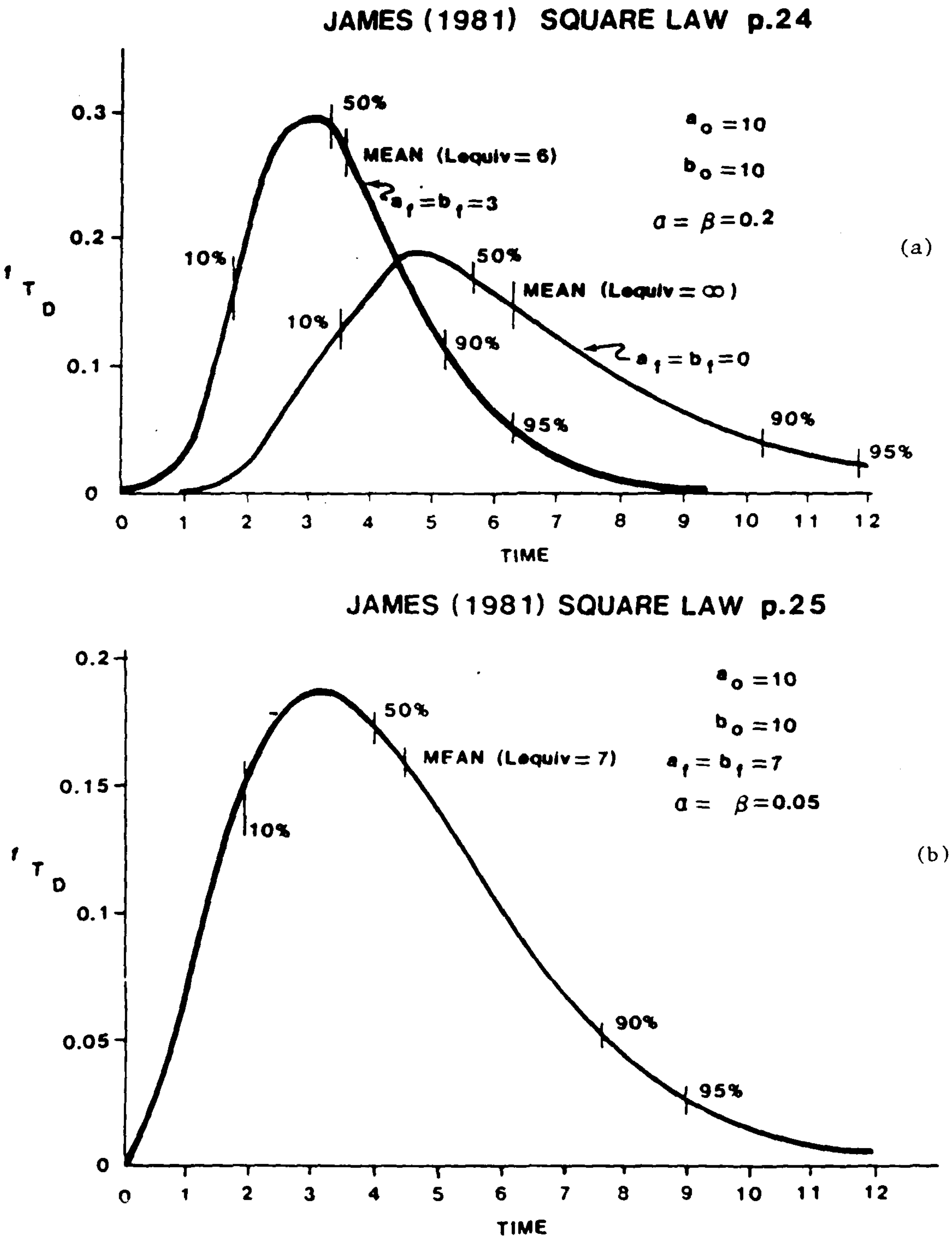


(a)



(b)

Figure III-2



JAMES (1981) SQUARE LAW p.24
$a_o = 10$
$b_o = 10$
$\alpha = \beta = 0.2$
(a)
50%
MEAN (Lequiv = 6)
$a_f = b_f = 3$
10%
50%
MEAN (Lequiv = ∞)
10%
90%
$a_f = b_f = 0$
95%
90%
95%
f_{T_D}
TIME
JAMES (1981) SQUARE LAW p.25
$a_o = 10$
$b_o = 10$
$a_f = b_f = 7$
$\alpha = \beta = 0.05$
(b)
50%
MFAN (Lequiv = 7)
10%
90%
95%
f_{T_D}
TIME

Figure III-3.

GYE AND LEWIS (1974) pp.23,24

SQUARE LAW

$a_0 = 50,\ b_0 = 100$

$a_f = b_f = 0$

$\alpha = \beta$

b	$\mu_{0,b}$	$\sigma^2_{0,b}$
100	0.404	0.00341
95	0.455	0.00394
90	0.509	0.00453
85	0.565	0.00519
90	0.626	0.00593
75	0.690	0.00679
70	0.752	0.00774
65	0.932	0.00836
60	0.911	0.01016

NOTE: $\mu_{0,b} = E[T_D \mid a(\infty) = b]$

$\sigma^2_{0,b} = V[T_D \mid a(\infty) = b]$

$E[T_D] = 0.552$

$V[T_D] = $ [illegible]

$t_f = 0.543$

(a)

JAMES (1981) p.37

SQUARE LAW

a_0	b_0	a_f	b_f	α	β	$E[T_D]$	$t_{.90}$	$t_{.95}$	t_f
10	10	0	0	0.20	0.20	6.36	10.0	12.0	∞
		3	3			3.56	5.5	6.5	6.92
		7	7	0.05	0.05	4.41	7.5	9.0	7.13
12	8	0	0	0.16	0.36	5.25	9.5	10.0	∞
		4	2	0.15	0.32	3.73	5.0	6.0	5.62
15	5	0	0	0.10	0.90	3.95	6.5	7.5	∞
		5	1	0.06	0.50	4.12	6.5	7.5	7.60
30	30	0	0	0.20	0.20	7.91	11.5	13.5	∞
		9	9			4.29	6.0	6.5	6.92
		21	21	0.05	0.05	5.42	7.5	8.5	7.13
45	15	0	0	0.10	0.90	[illegible]	7.5	9.5	∞
		15	3	0.06	0.50	5.09	7.0	7.5	7.60
10	10	0	0	0.30	0.20	4.66	7.5	9.0	4.69
		3	3			2.70	4.5	5.0	3.90
		7	7	0.075	0.05	3.46	6.0	7.0	4.44
15	5	0	0	0.15	0.90	3.04	5.5	6.0	3.12
		5	1	0.09	0.50	3.29	5.5	6.0	3.91
30	30	0	0	0.30	0.20	5.09	7.5	9.0	4.69
		9	9			3.02	4.5	4.5	3.90
		21	21	0.075	0.50	4.13	6.0	6.5	4.44
45	15	0	0	0.15	0.90	3.41	5.5	6.5	3.12
		15	3	0.90	0.50	3.79	5.5	6.0	3.31
20	5	0	0	0.10	0.90	3.50	6.0	7.5	3.74
		6	1	0.60	0.50	3.96	7.0	8.0	4.07
60	15	0	0	0.10	0.90	3.56	5.5	6.5	3.74
		13	3	0.60	0.50	4.23	6.5	7.5	4.07

(b)

SPRINGALL (1968) p.54

SQUARE LAW

$a_0 = 10,\ b_0 = 8,\ \alpha = 1,\ \beta = 1$

$a_f = b_f = 0$

N.B.: $t_f = 1.098$

$E(T_D) = 1.1070$ time units, $E(T_D^2) = 1.5412\ (\text{t.u.})^2$,

$E(T_D^3) = 2.6737\ (\text{t.u.})^3$, $E(T_D^4) = 5.6953\ (\text{t.u.})^4$,

Variance of distribution of duration = 0.31533 (t.u.)^2,

Skewness of distribution of duration = 2.25632,

Kurtosis of distribution of duration = 6.35475.

(c)

Table III-1

WEALE (1976) pp.43-45

SQUARE LAW (WITH DRAWS)

$a_0 = b_0 = 10$

$a_f = 2,\ b_f = 3$

$\alpha = 0.16,\ \beta = 0.08$

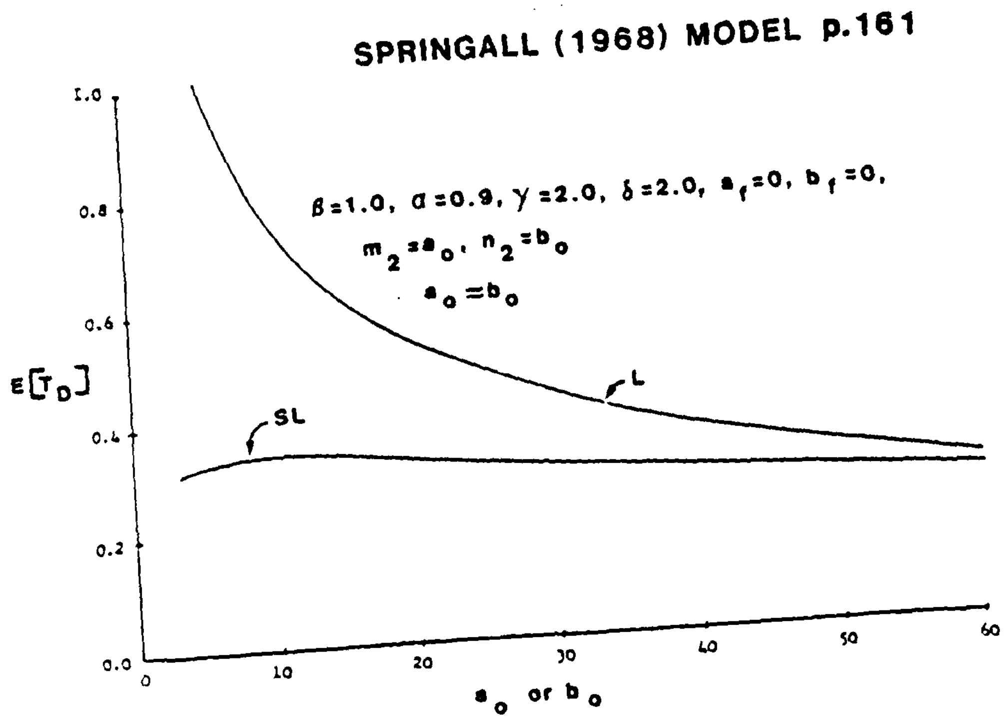
$P(A) = 0.77630301,\ P(B) = 0.03090876,\ P(D) = 0.19278822$

DRAWS OCCUR IF:

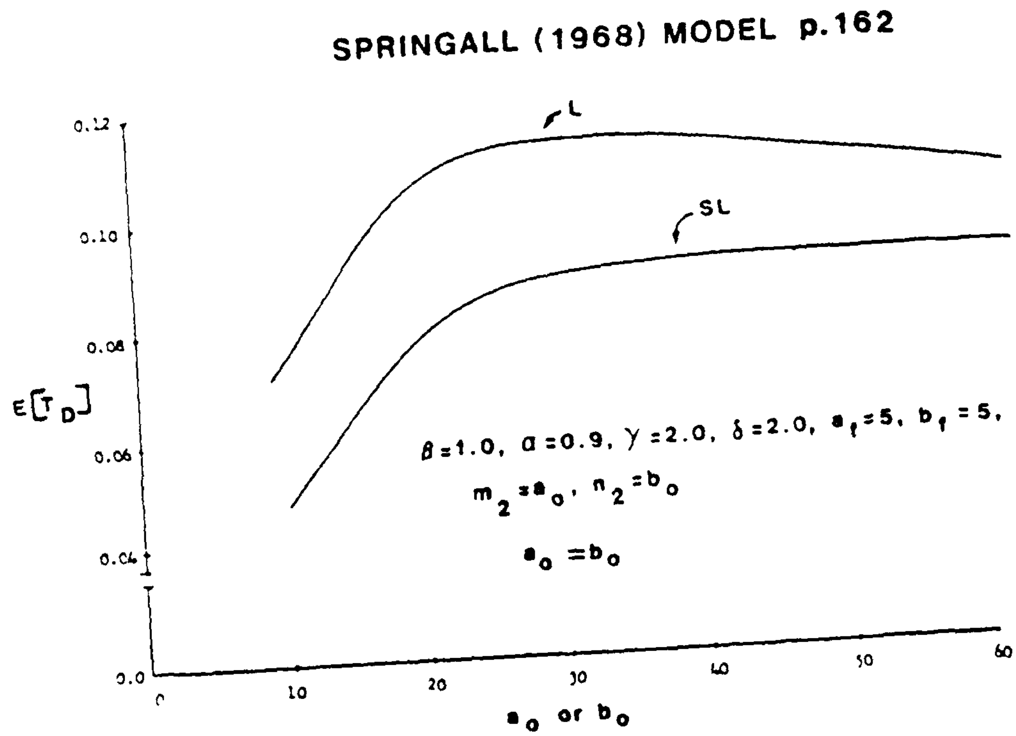
$a = 2,\ b \leq 6$ or
$b = 3,\ a \leq 5$

T	$E[T_D] = 5.35$		$E[T_{D\|A}] = 4.58$		$E[T_{D\|B}] = 6.21$		$E[T_{D\|D}] = 8.31$	
	F_{T_D}	f_{T_D}	$F_{T_{D\|A}}$	$f_{T_{D\|A}}$	$F_{T_{D\|B}}$	$f_{T_{D\|B}}$	$F_{T_{D\|D}}$	$f_{T_{D\|D}}$
0.00	0.00000	0.000	0.00000	0.000	0.00000	0.000	0.00000	0.000
2.00	0.03466	0.073	0.04445	0.093	0.00419	0.012	0.00010	0.000
4.00	0.33787	0.193	0.42414	0.234	0.14074	0.140	0.02210	0.035
6.00	0.67320	0.130	0.79953	0.127	0.50572	0.192	0.19136	0.134
8.00	0.86077	0.063	0.95205	0.038	0.81202	0.107	0.50103	0.157
10.00	0.94620	0.027	0.99118	0.008	0.94756	0.037	0.76484	0.102
12.00	0.98115	0.010	0.99866	0.001	0.98824	0.009	0.90952	0.047
14.00	0.99400	0.004	0.99982	0.000	0.99774	0.002	0.96995	0.017
16.00	0.99825	0.001	0.99998	0.000	0.99961	0.000	0.99104	0.006
18.00	0.99952	0.000	1.00000	0.000	0.99994	0.000	0.99754	0.002
20.00	0.99988	0.000	1.00000	0.000	0.99999	0.000	0.99936	0.000
22.00	0.99997	0.000	1.00000	0.000	1.00000	0.000	0.99984	0.000
24.00	0.99999	0.000	1.00000	0.000	1.00000	0.000	0.99996	0.000
26.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	0.99999	0.000
28.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	1.00000	0.000
30.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	1.00000	0.000

Table III-2.



(a)



(b)

Figure III-4.

WEALE (1976) pp.15,16,17

SPECIAL MODEL

$a_0 = b_0 = 10$

$a_f = b_f = 3$

$P(A) = .99280858,\ P(B) = .00001826,\ P(D) = .00717316$

DRAWS OCCUR IF:

$a = 2,\ b \le 6$ or
$b = 3,\ a \le 5$

ATTRITION FUNCTIONS:

A Side = a(0.05 + 0.05b)
B Side = b(0.025 + 0.005a)

T	$E[T_D] = 2.02$		$E[T_{D\|A}] = 2.00$		$E[T_{D\|B}] = 3.43$		$E[T_{D\|D}] = 4.44$	
	F_{T_D}	f_{T_D}	$F_{T_{D\|A}}$	$f_{T_{D\|A}}$	$F_{T_{D\|B}}$	$f_{T_{D\|B}}$	$F_{T_{D\|D}}$	$f_{T_{D\|D}}$
0.00	0.00000	0.000	0.00000	0.000	0.00000	0.000	0.00000	0.000
1.00	0.08261	0.314	0.08320	0.316	0.00235	0.015	0.00009	0.001
2.00	0.56262	0.471	0.56656	0.474	0.09754	0.210	0.01865	0.059
3.00	0.87641	0.172	0.88158	0.171	0.40279	0.353	0.16248	0.231
4.00	0.97035	0.043	0.97420	0.041	0.71639	0.250	0.43818	0.290
5.00	0.99267	0.010	0.99483	0.008	0.89422	0.114	0.69471	0.211
6.00	0.99797	0.002	0.99900	0.002	0.96638	0.041	0.85568	0.115
7.00	0.99936	0.001	0.99981	0.000	0.99038	0.012	0.93695	0.054
8.00	0.99977	0.000	0.99996	0.000	0.99743	0.003	0.97350	0.023
9.00	0.99991	0.000	0.99999	0.000	0.99934	0.001	0.98904	0.010
10.00	0.99997	0.000	1.00000	0.000	0.99984	0.000	0.99549	0.004
11.00	0.99999	0.000	1.00000	0.000	0.99996	0.000	0.99814	0.002
12.00	0.99999	0.000	1.00000	0.000	0.99999	0.000	0.99924	0.001
13.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	0.99969	0.000
14.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	0.99987	0.000
15.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	0.99995	0.000
16.00	1.00000	0.000	1.00000	0.000	1.00000	0.000	0.99998	0.000

(a)

SPRINGALL (1968) p.177

SPRINGALL MODEL

$a_0 = 60,\ b_0 = 60,\ \alpha = 0.9,\ \beta = 1.0,\ \gamma = 2.0,\ \delta = 2.0$

$a_f = b_f = 5$

m_2 and n_2	$E[T_D]$
10	0.45087
15	0.23268
20	0.15671
30	0.10796
40	0.09529
50	0.09165
60	0.09091

(b)

Table III-3.

IV. THE PROBABILITY OF WINNING

We note again that, for L parity, the L prediction is that both sides are annihilated or both sides reach their breakpoints. In either case, neither side wins and therefore L predicts P(A) = P(B) = 0. At parity, the L probabilities have a discontinuity and make a jump with a predicted probability of zero at the parity point. At the corresponding SL point, the win probabilities are P(A) = P(B) = 1/2 for strict L parity and will be some value different from 1/2 for non-strict L parity. In the latter case, it may even be on the wrong side of 1/2, e.g., SL P(A) may be greater than 1/2 when L P(A) = 0. The jump points are shown as dashed vertical lines on almost all curves or are otherwise indicated on some curves and all tables except Figure IV-15 and Table IV-10 where no L information is given.

Some other sections in Part Two contain figures and tables on the probability of winning and should be consulted. These are:

(1) The Square Law; Figures I-1(b), (c), I-2, I-6, I-7, I-8, III-1, III-2 (in these latter two figures $F_{T_D}(\infty)$ are P(A), P(B) on the A, B victory curves respectively), V-12, V-13, V-14 and Tables II-1, II-2, and V-9.

(2) The Square Law with draws; Table III-2

(3) The Linear Law; Tables I-5(a) and II-3.

(4) The Springall Model; Tables II-4(c), V-6 and V-11.

(5) The Weale Special Model; Table III-3(a) and V-10.

Some of these contain L comparisions and some do not. Usually, the L predictions are obvious or easily determined from the discussion in Part One.

The probability of winning figures all are shown as continuous functions of the parameters a_0, b_0 (also a_f and b_f in some cases), α, β (which may be exhibited in terms of P_A, P_B, μ_A, and μ_B). If a_0 and b_0 are fixed, and if α (or β) or some function of α and β is varied the curves are genuinely continuous. However, this is the case only for Figures IV-11, IV-10(b) (showing slope discontinuities which it should not) and IV-17. All others should have values at certain discontinuous points only. In other words, if a_0 or b_0 is the variable, the P(A) and P(B) curves should be discontinuous. This can be somewhat misleading and care should be taken in reading the curves. Figure IV-2 illustrates this point. α and β are fixed, $a_f = b_f = 0$ and the curve parameters are $a_0 + b_0$. If, for example, $a_0 + b_0 = 5$, a_0 can only assume values 0,1,2,3,4, and 5 while the corresponding values of b_0 are 5,4,3,2,1, and 0. P(B) only has non-zero values at abscissa points 0, 1/16, 4/9, 9/4, 4 and ∞ . Interestingly, there is only one value to the right of the jump point. Also one has to be curious about the curve whose parameter is $a_0 + b_0 = 0$ as this implies neither side has any combatants! If the authors used some limiting process to obtain this curve or assumed a_0, b_0 continuous for the others, they do not say so.

In any event, the presented material contains a wealth of evidence that L is a very poor predictor of the SL probability of winning.

GYE & LEWIS (1974, 1976) SQUARE LAW p.21 (1974)
p.117 (1976)

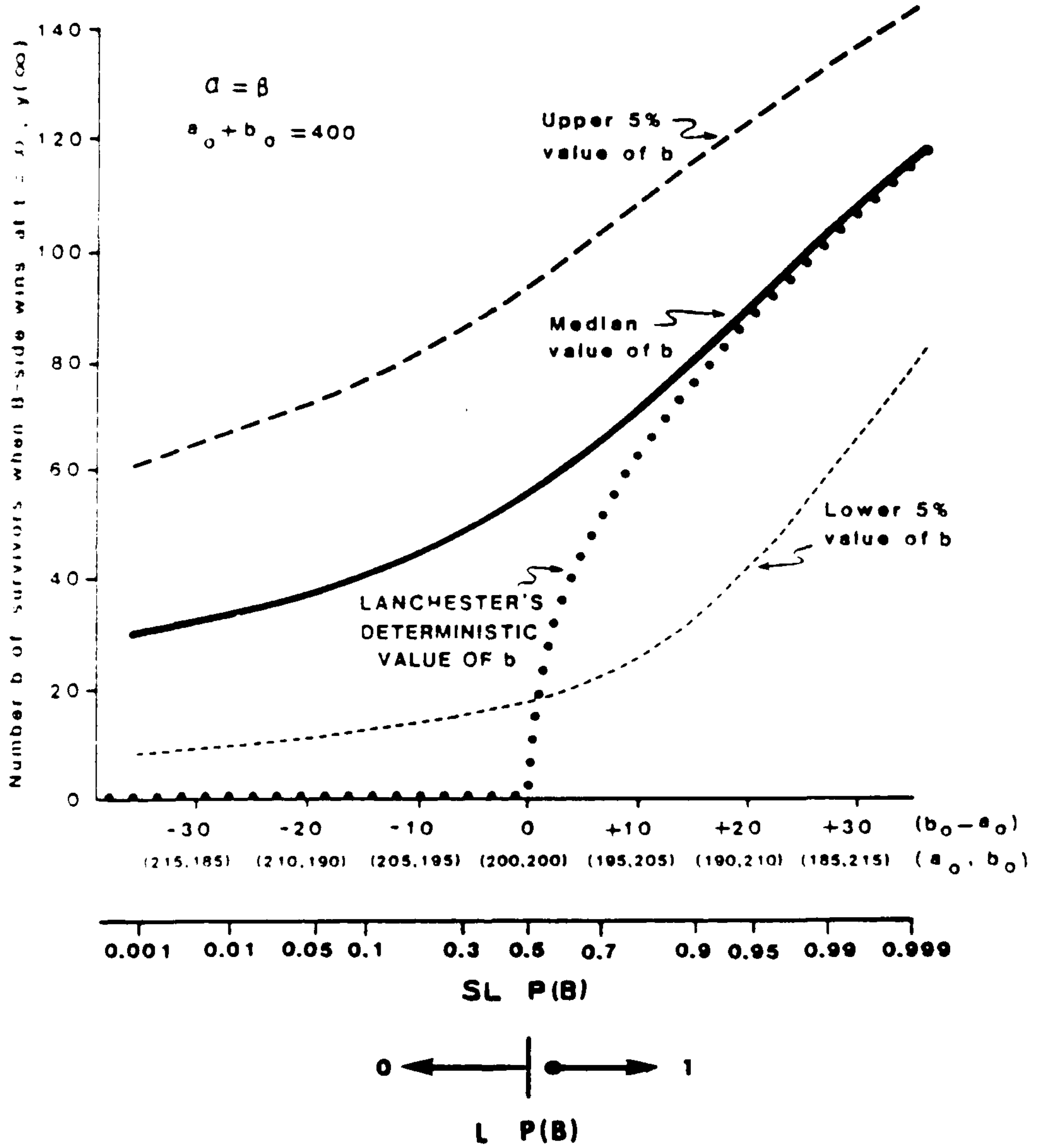


Figure IV-1.

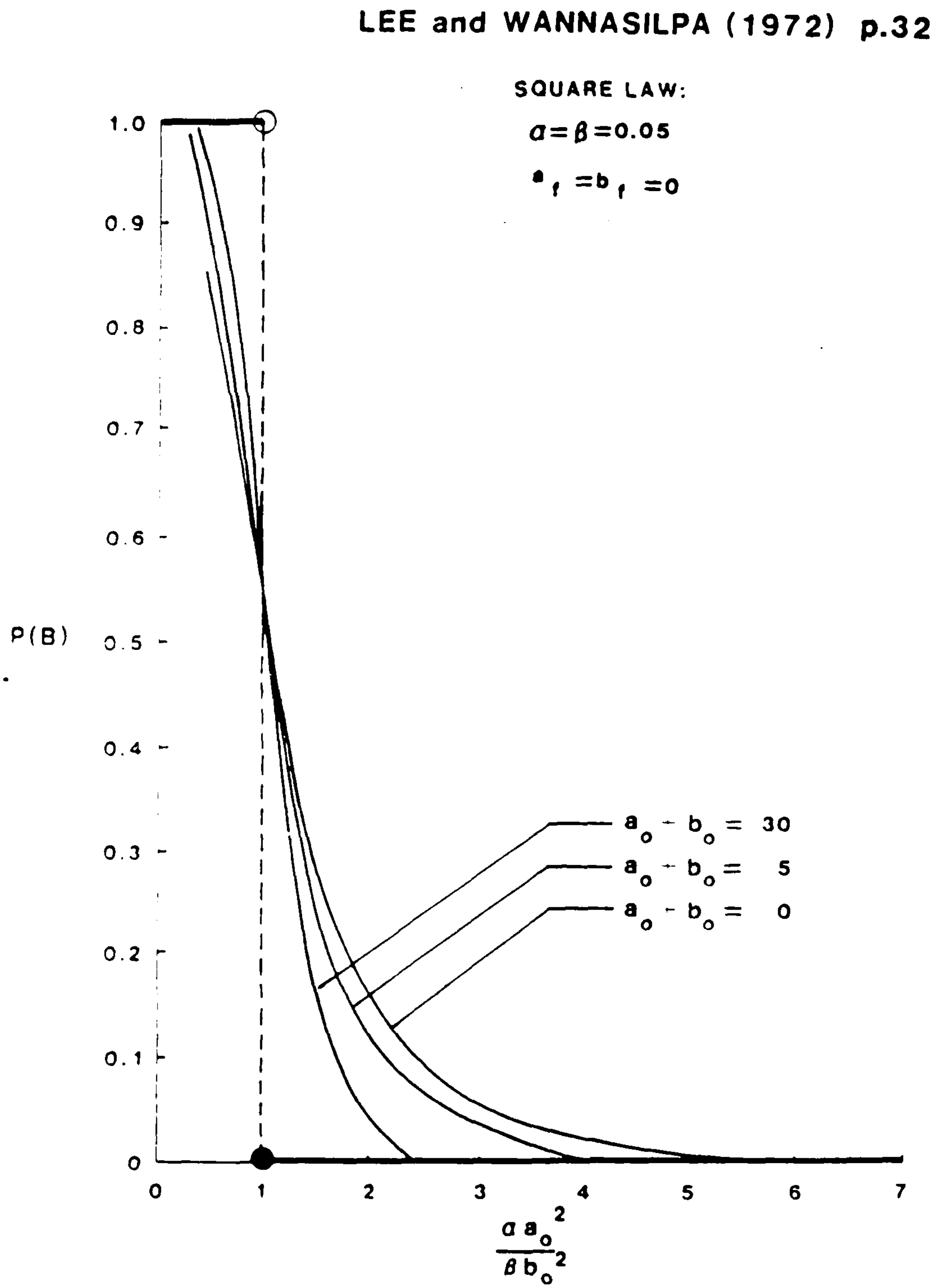
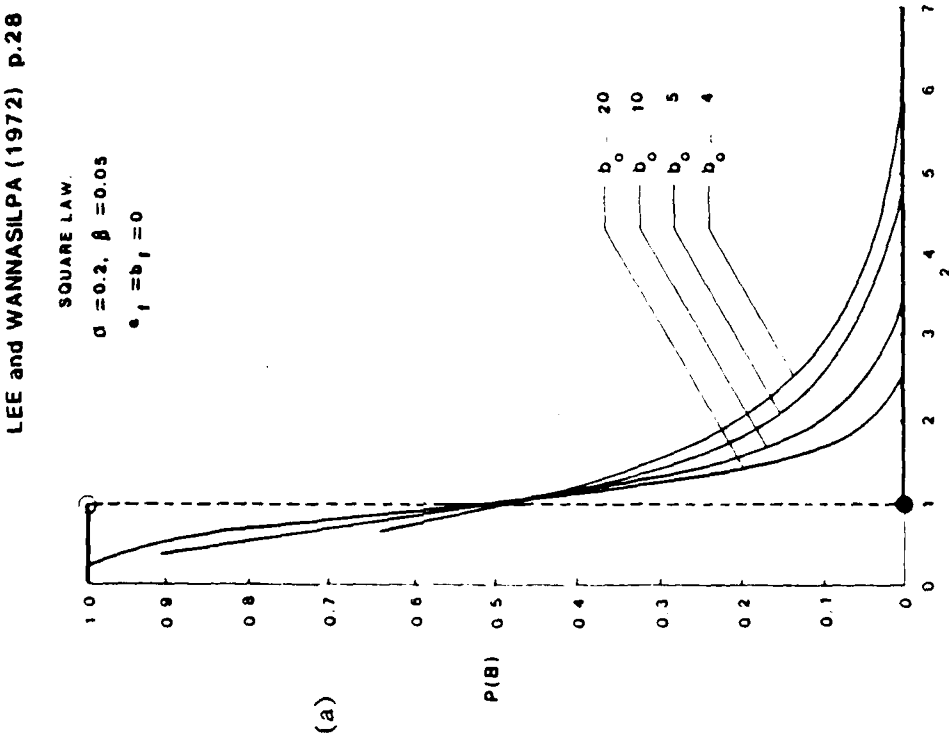

LEE and WANNASILPA (1972) p.32
SQUARE LAW:
$\alpha=\beta=0.05$
$a_f = b_f = 0$
P(B)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
$a_o - b_o = 30$
$a_o - b_o = 5$
$a_o - b_o = 0$
$\frac{\alpha a_o^2}{\beta b_o^2}$

Figure IV-2


LEE and WANNASILPA (1972) p.28
SQUARE LAW
α = 0.2, β = 0.05
a1 = b1 = 0
P(B)
(a)

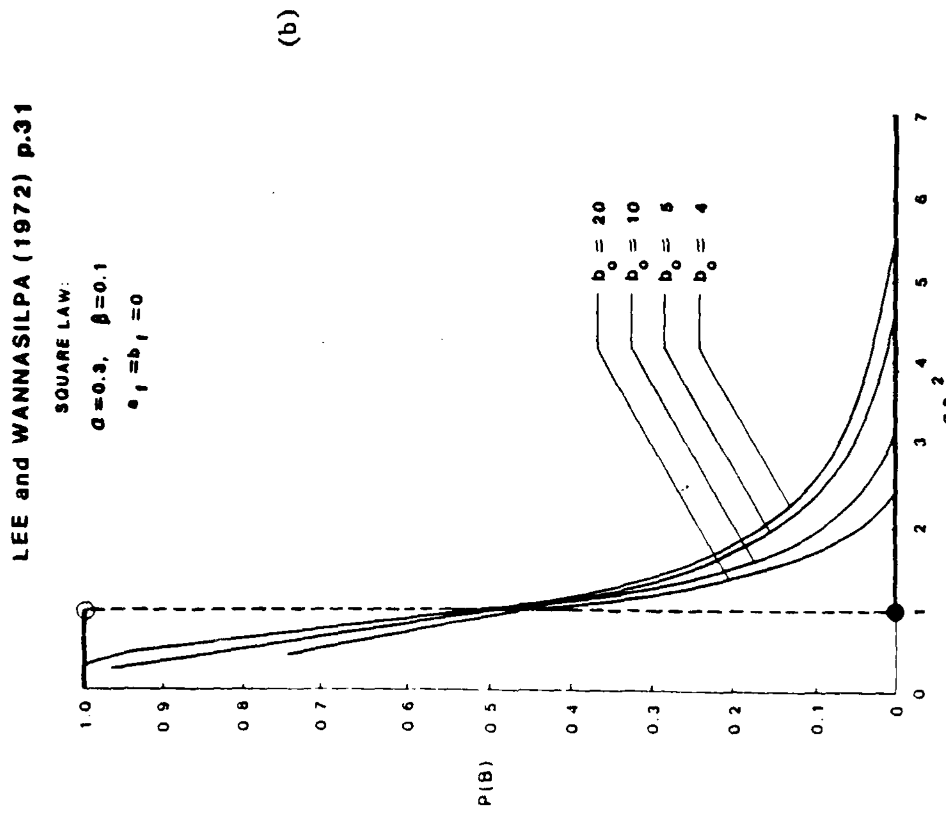

LEE and WANNASILPA (1972) p.31
SQUARE LAW:
α = 0.3, β = 0.1
a1 = b1 = 0
bo = 20
bo = 10
bo = 5
bo = 4
P(B)
(b)

Figure IV-3

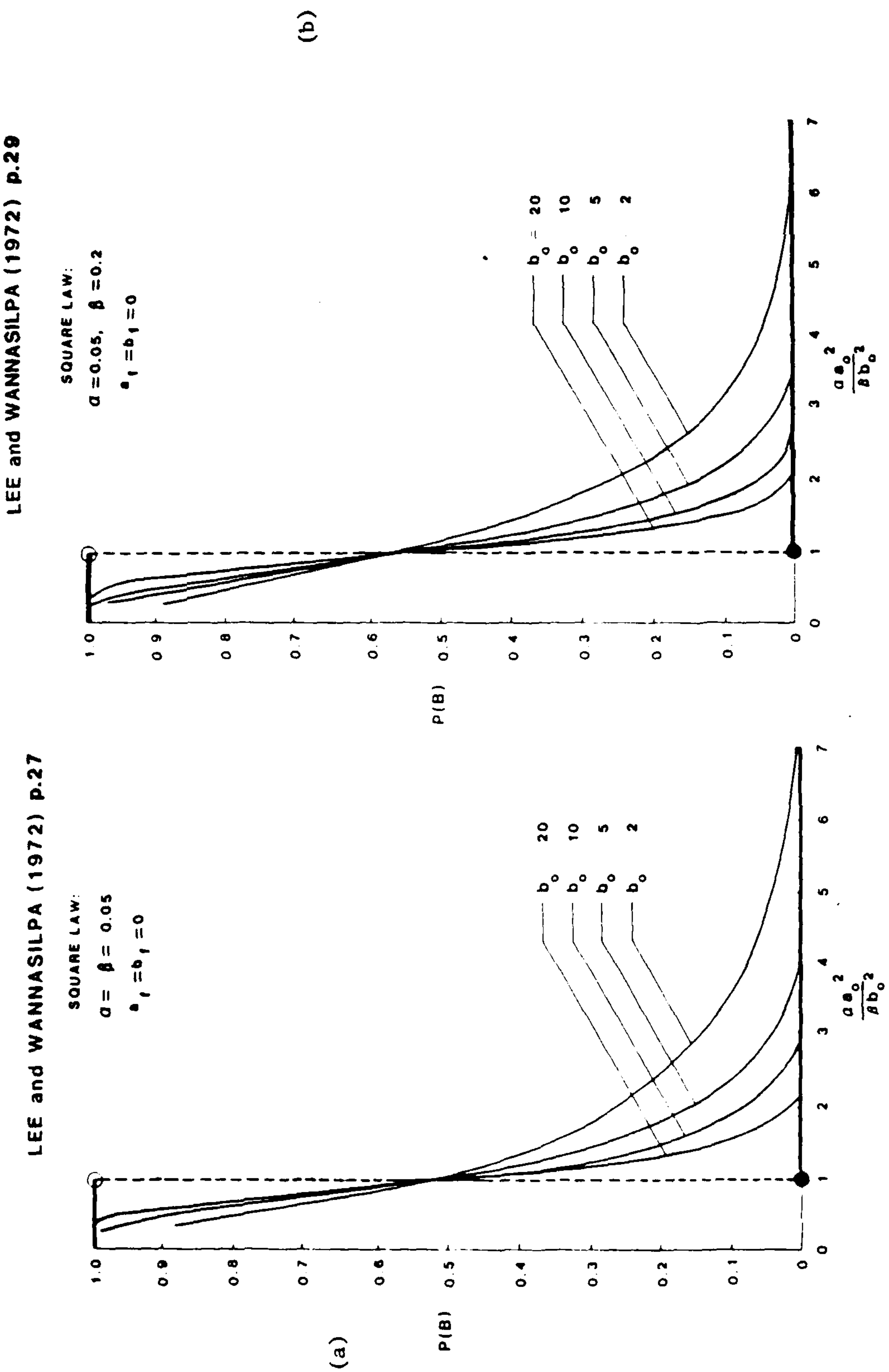

(a)
LEE and WANNASILPA (1972) p.27
SQUARE LAW:
$\alpha = \beta = 0.05$
$a_1 = b_1 = 0$
b_o 20
b_o 10
b_o 5
b_o 2
P(B)
$\frac{\alpha a_o^2}{\beta b_o^2}$
(b)
LEE and WANNASILPA (1972) p.29
SQUARE LAW:
$\alpha = 0.05, \ \beta = 0.2$
$a_1 = b_1 = 0$
b_o = 20
b_o 10
b_o 5
b_o 2
P(B)
$\frac{\alpha a_o^2}{\beta b_o^2}$

Figure IV-4.

LEE and WANNASILPA (1972) p.30

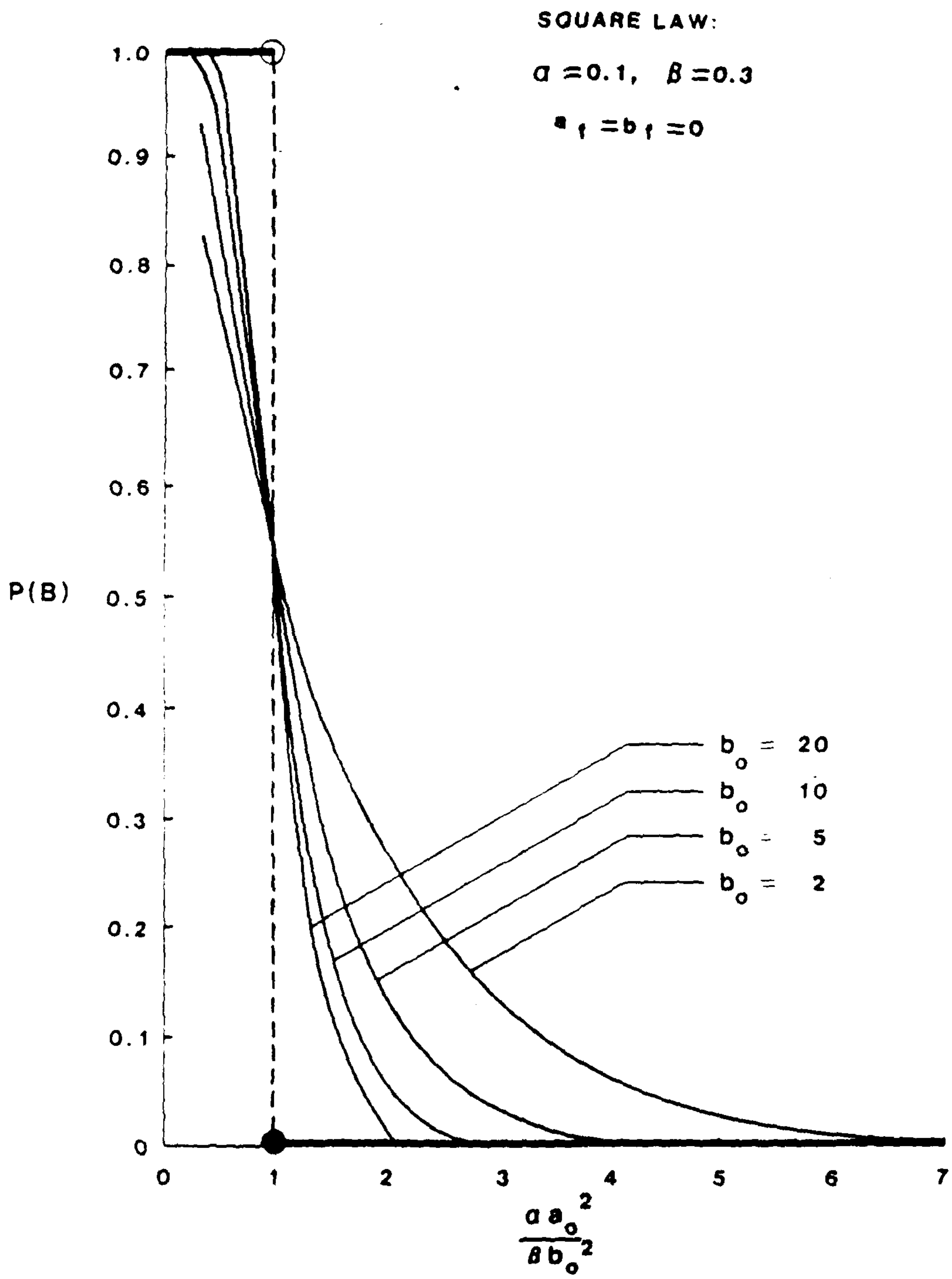


Figure IV-5.

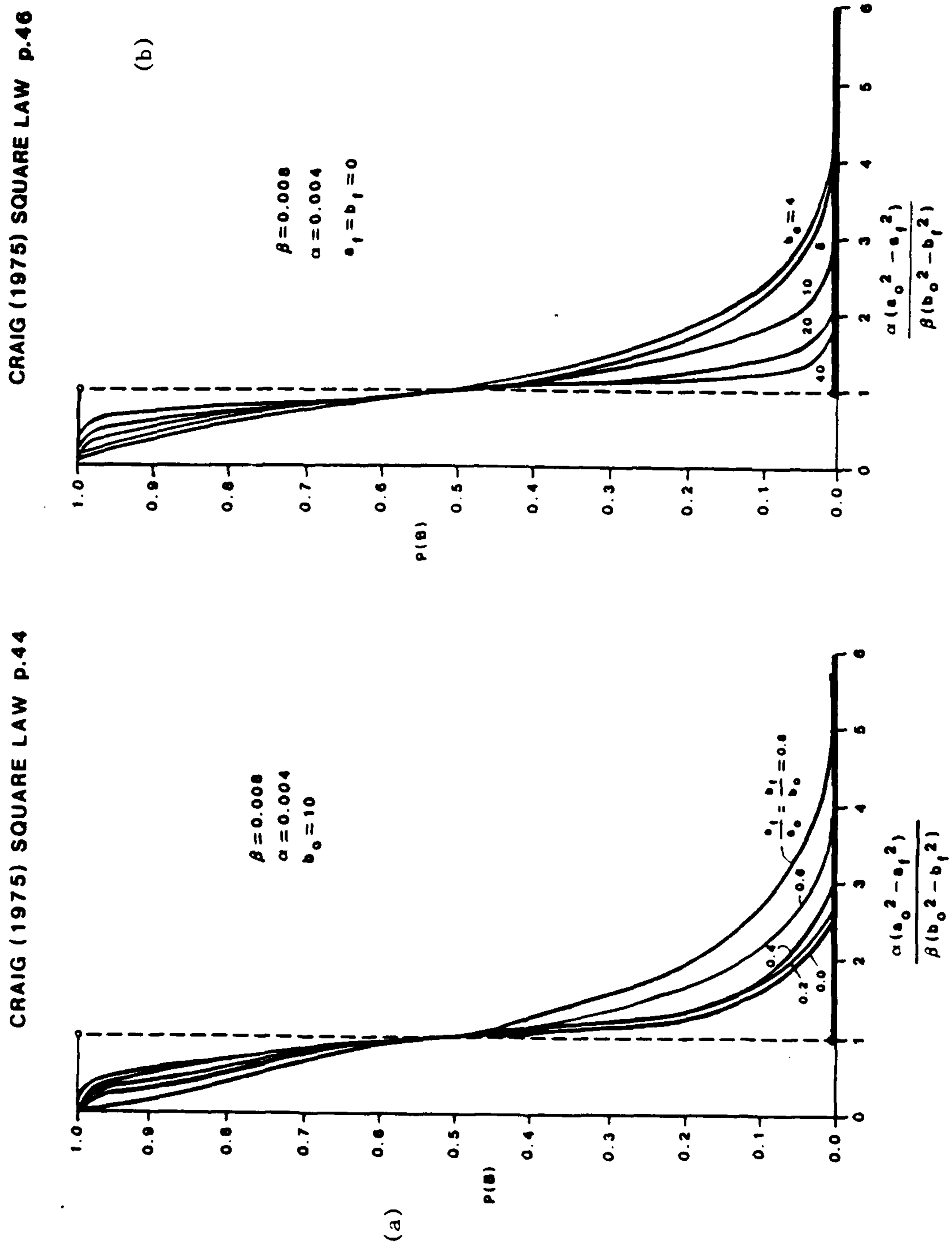

CRAIG (1975) SQUARE LAW p.46
(b)
β=0.008
α=0.004
a_f=b_f=0
b_o=4
8
10
20
40
P(B)
α(a_o^2−a_f^2) / β(b_o^2−b_f^2)
CRAIG (1975) SQUARE LAW p.44
(a)
β=0.008
α=0.004
b_o=10
a_f/a_o = b_f/b_o = 0.8
0.6
0.4
0.2
0.0
P(B)
α(a_o^2−a_f^2) / β(b_o^2−b_f^2)

Figure IV-6

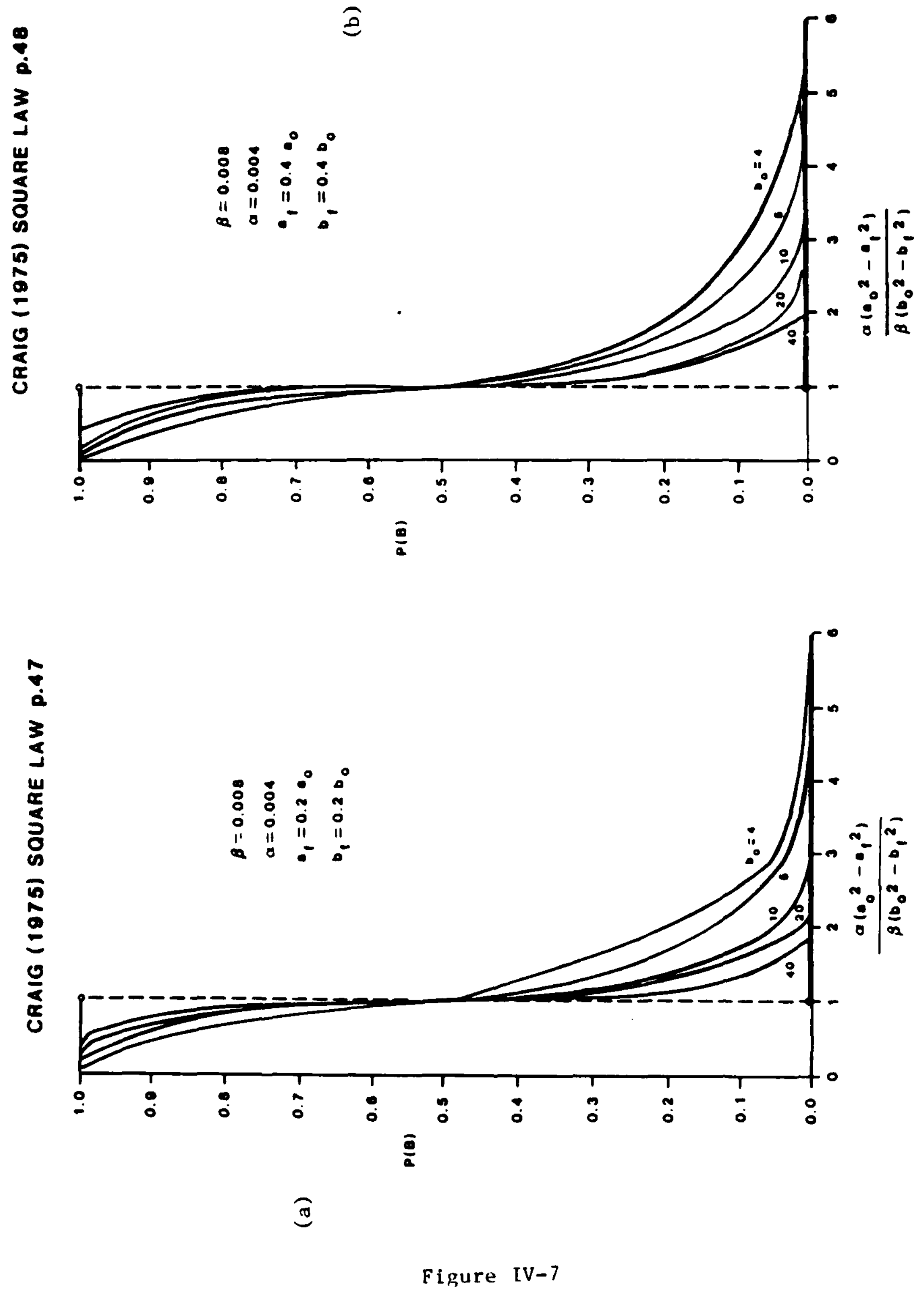

CRAIG (1975) SQUARE LAW p.47
CRAIG (1975) SQUARE LAW p.48
(a)
(b)
β = 0.008
α = 0.004
a_f = 0.2 a_o
b_f = 0.2 b_o
a_f = 0.4 a_o
b_f = 0.4 b_o
P(B)
b_o = 4
6
10
20
40
α(a_o² − a_f²) / β(b_o² − b_f²)

Figure IV-7

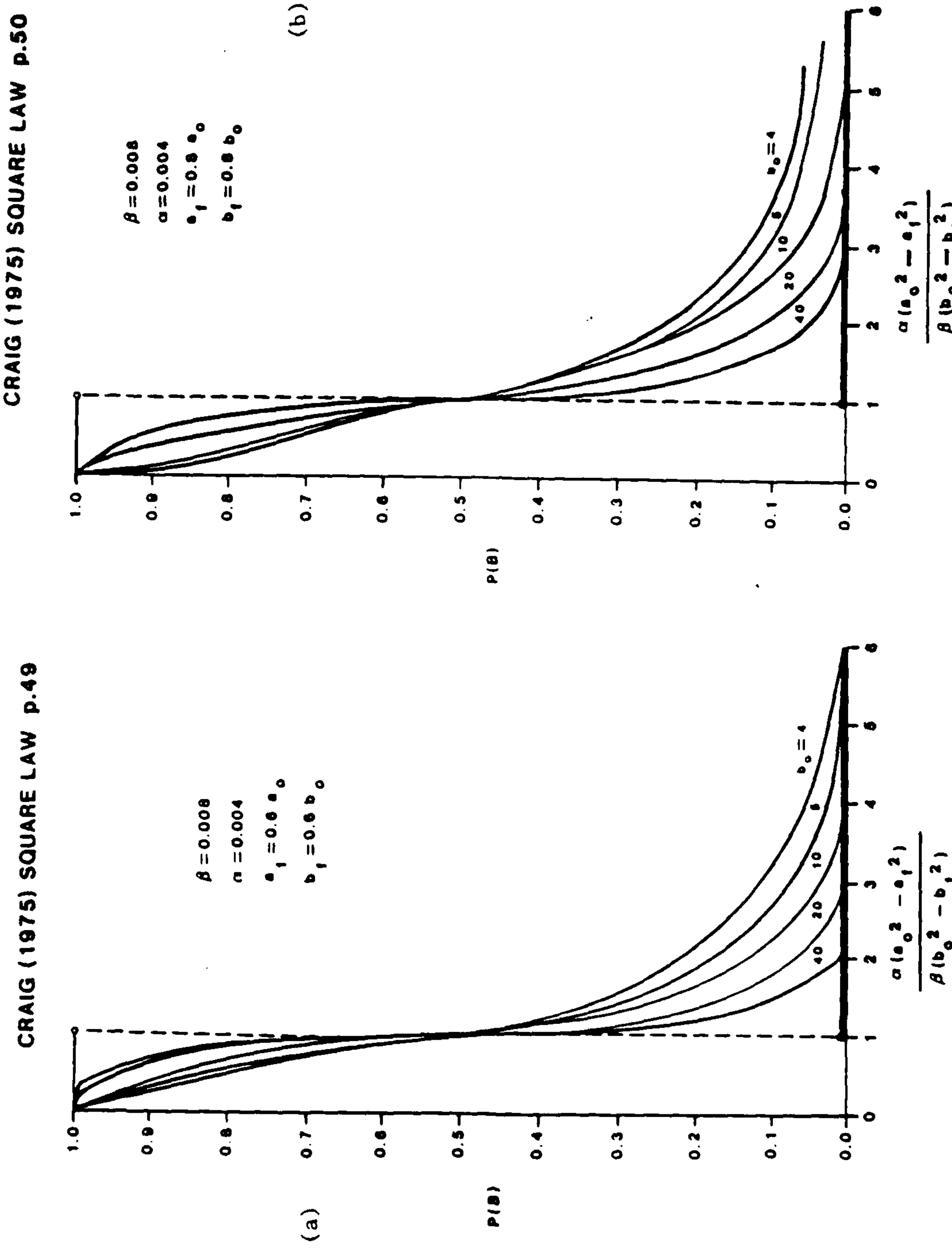

(a)
CRAIG (1975) SQUARE LAW p.49
β = 0.008
α = 0.004
a_f = 0.6 a_o
b_f = 0.6 b_o
P(B)
b_o = 4
6
10
20
40
α(a_o² − a_f²) / β(b_o² − b_f²)
(b)
CRAIG (1975) SQUARE LAW p.50
β = 0.008
α = 0.004
a_f = 0.8 a_o
b_f = 0.8 b_o
P(B)
b_o = 4
6
10
20
40
α(a_o² − a_f²) / β(b_o² − b_f²)

Figure IV-8

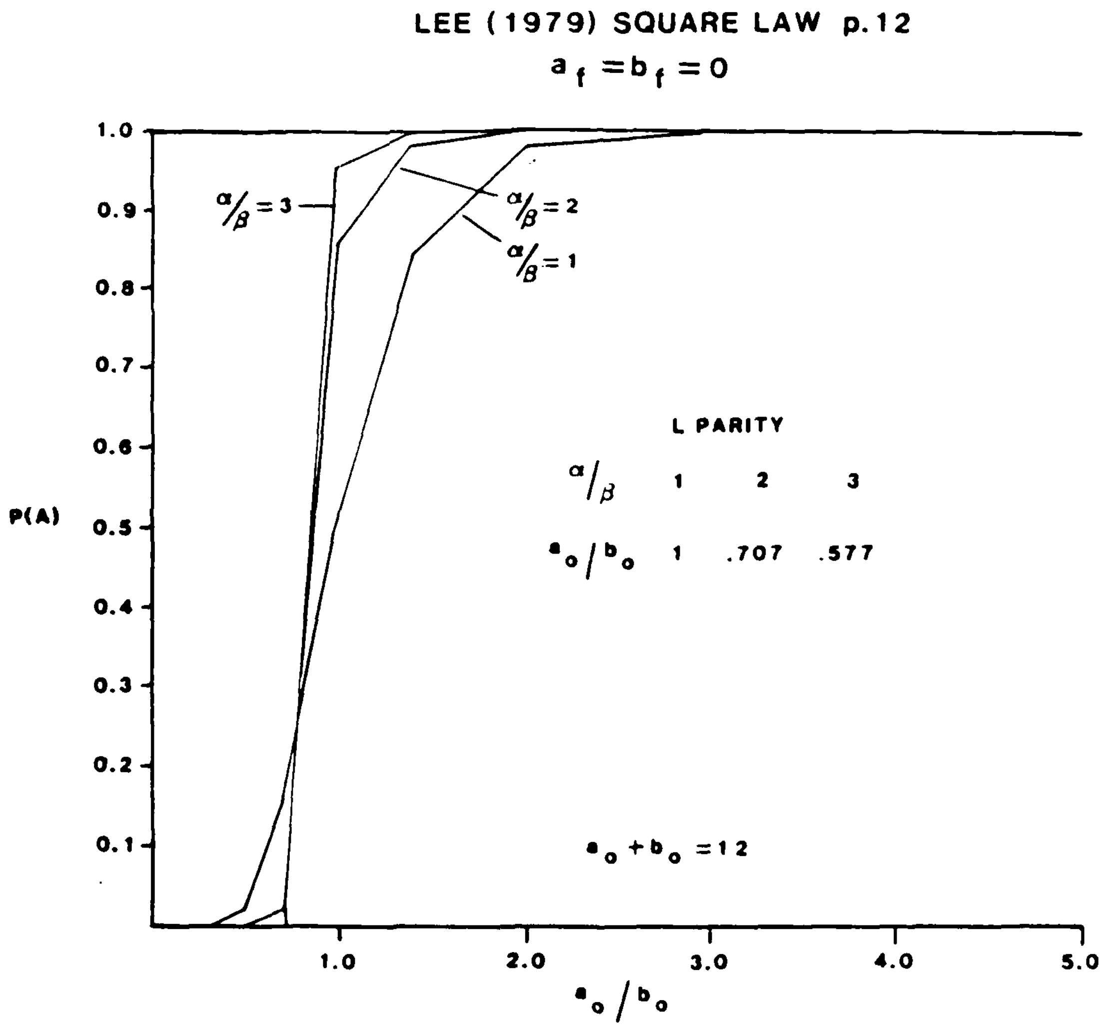
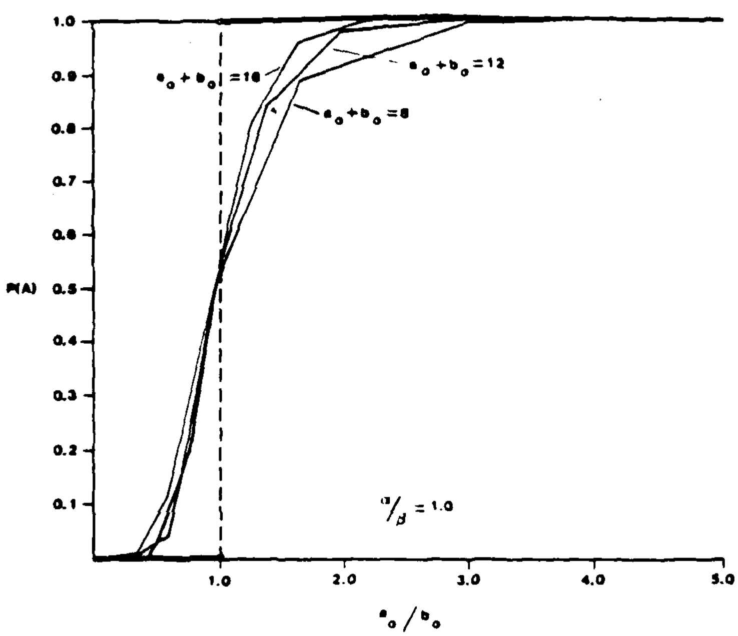
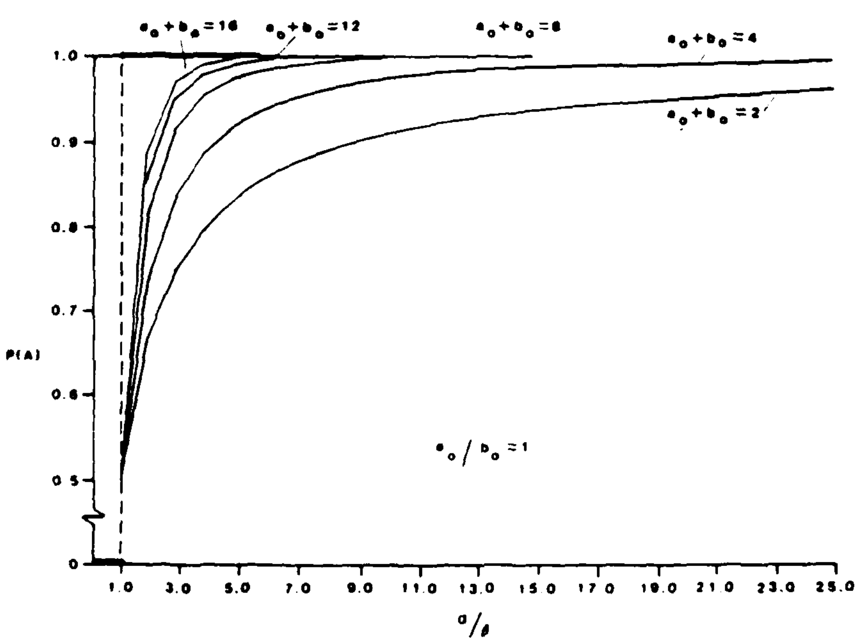

LEE (1979) SQUARE LAW p.12
$a_f = b_f = 0$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
P(A)
$\alpha/\beta = 3$
$\alpha/\beta = 2$
$\alpha/\beta = 1$
L PARITY
α/β 1 2 3
a_o/b_o 1 .707 .577
$a_o + b_o = 12$
1.0
2.0
3.0
4.0
5.0
a_o/b_o

Figure IV-9



(a)



(b)

Figure IV-10

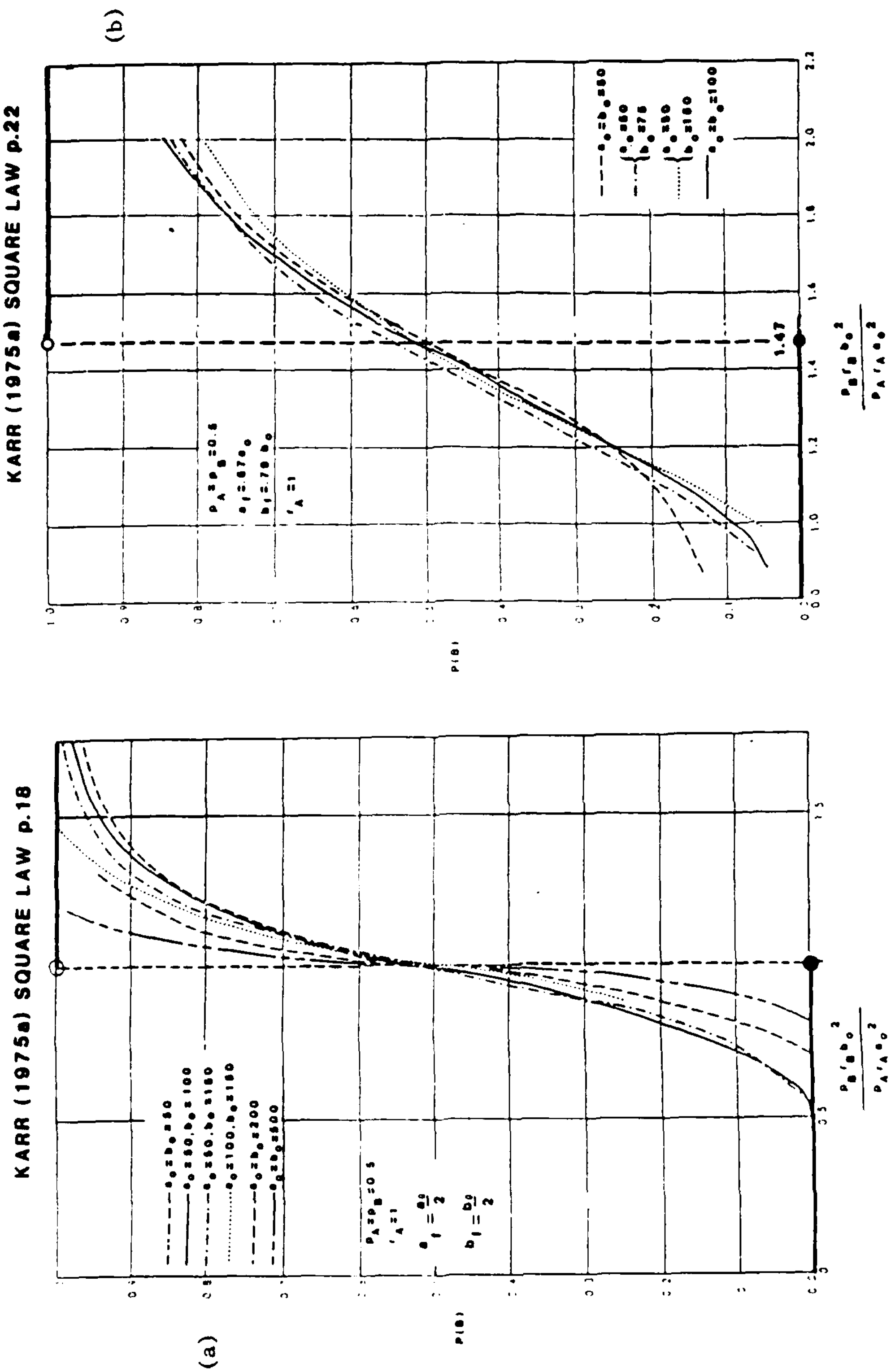
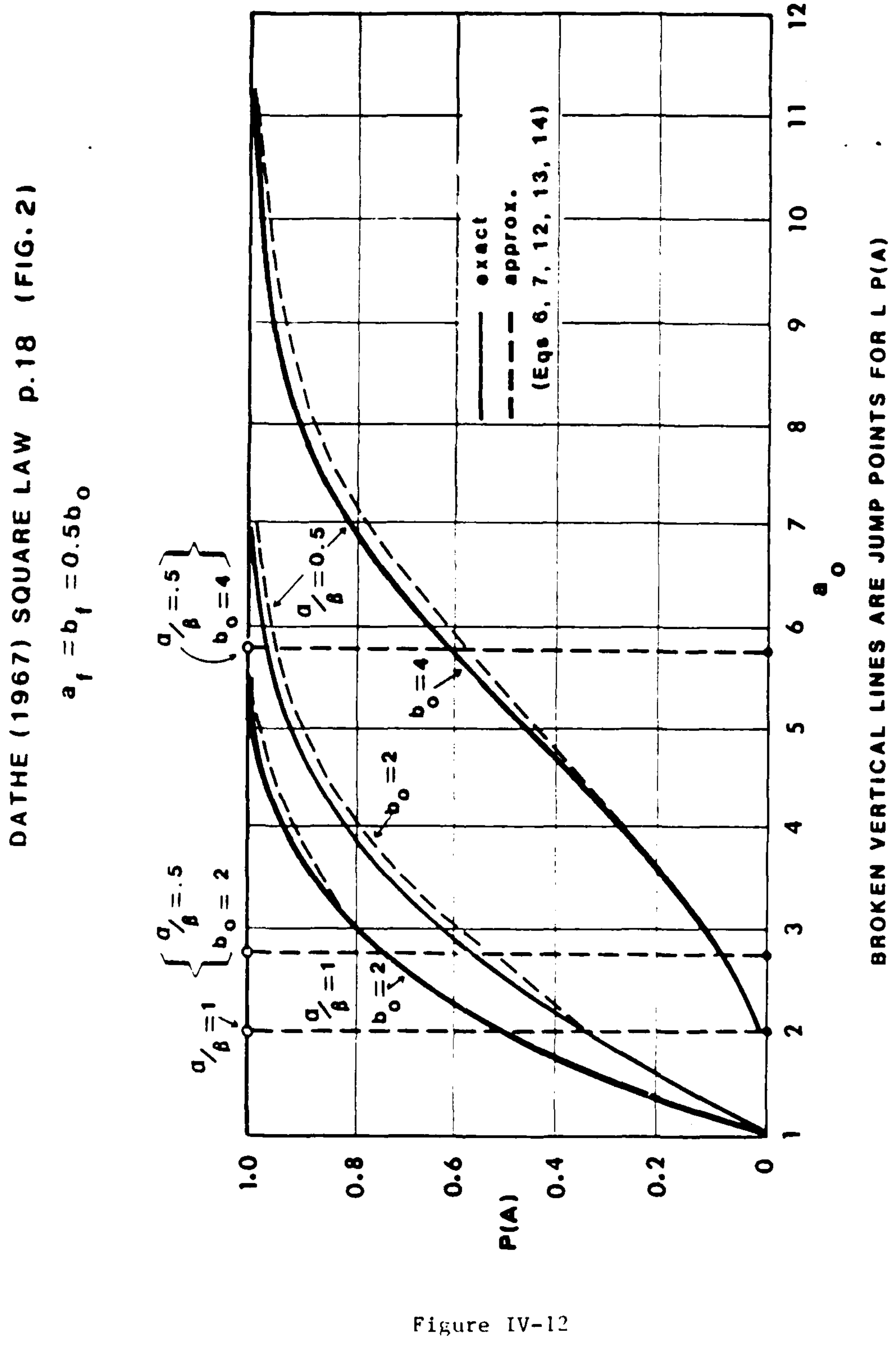
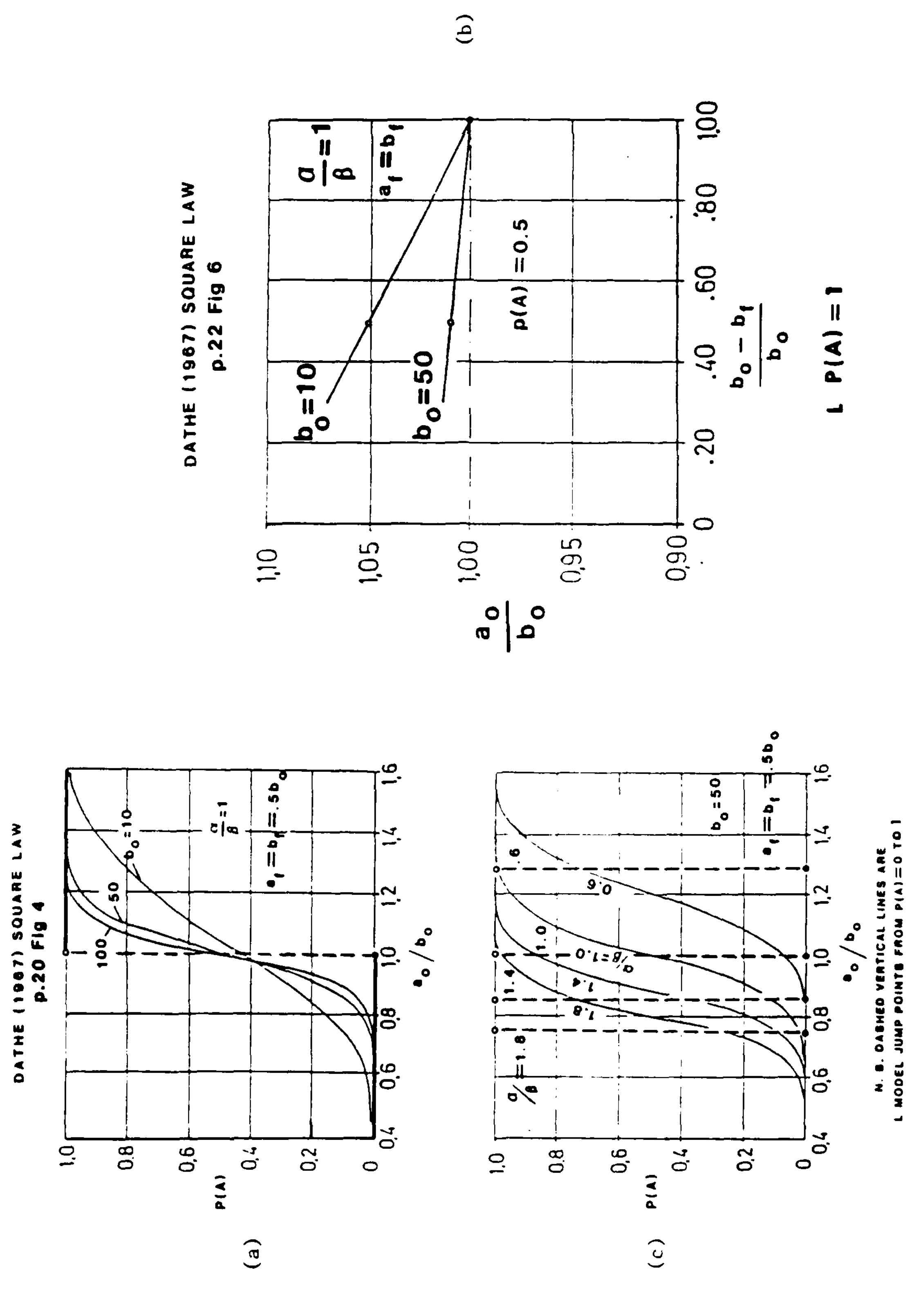


Figure IV-11



DATHE (1967) SQUARE LAW p.18 (FIG. 2)
$a_f = b_f = 0.5 b_o$
P(A)
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10
11
12
a_o
$\alpha/\beta = 1$
$\alpha/\beta = .5$, $b_o = 2$
$\alpha/\beta = .5$, $b_o = 4$
$\alpha/\beta = 1$
$b_o = 2$
$b_o = 2$
$b_o = 4$
$\alpha/\beta = 0.5$
exact
approx.
(Eqs 6, 7, 12, 13, 14)
BROKEN VERTICAL LINES ARE JUMP POINTS FOR L P(A)

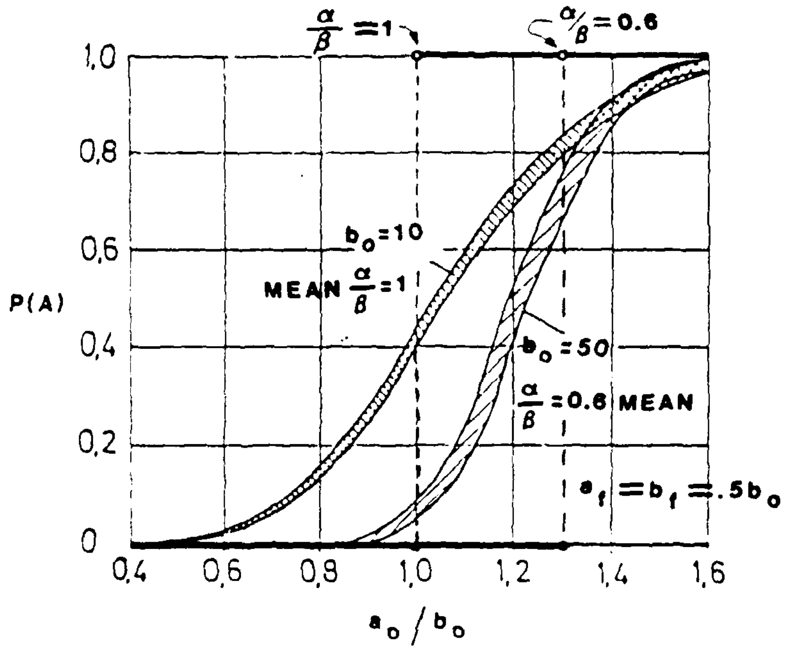
Figure IV-12



DATHE (1967) SQUARE LAW
p.22 Fig 6
$b_o=10$
$b_o=50$
$\frac{\alpha}{\beta}=1$
$a_f=b_f$
p(A) = 0.5
$\frac{a_o}{b_o}$
$\frac{b_o-b_f}{b_o}$
L P(A) = 1
(b)
DATHE (1967) SQUARE LAW
p.20 Fig 4
$b_o=10$
50
100
$\frac{\alpha}{\beta}=1$
$a_f=b_f=.5b_o$
P(A)
a_o/b_o
(a)
$b_o=50$
$a_f=b_f=.5b_o$
$\alpha/\beta=1.8$
1.4
1.0
.6
P(A)
a_o/b_o
N. B. DASHED VERTICAL LINES ARE
L MODEL JUMP POINTS FROM P(A)=0 TO 1
(c)

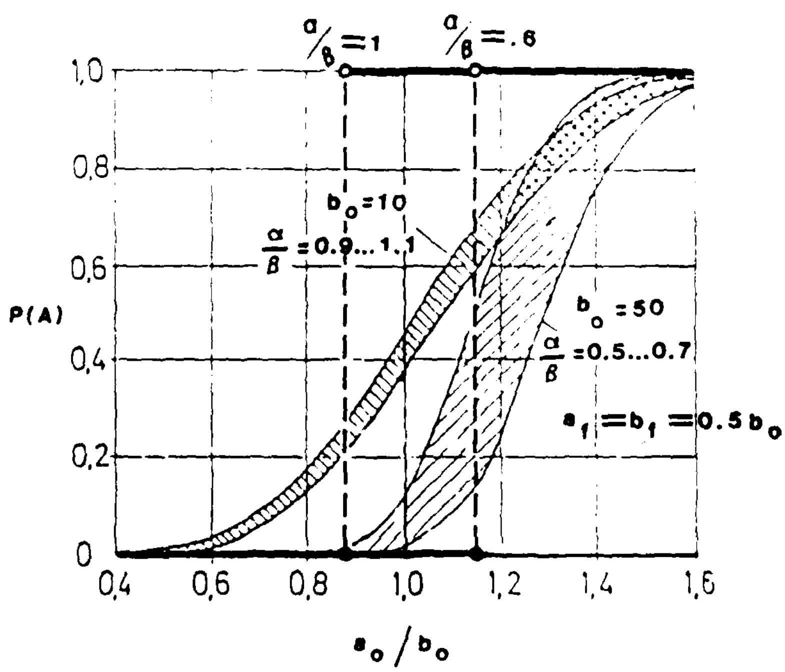
Figure IV-13

DATHE (1967) SQUARE LAW p.21, Fig 5



(a)

95% CONFIDENCE INTERVAL ON α/β



(b)

α/β VARIES BY ± 0.1

Figure IV-14

(a)

BROWN (1955) p.29

SQUARE LAW

$a_f = b_f = 0$

$\alpha = \beta$

P(A)

a_0 \ b_0	0	1	2	3	4	5
0	-	0.00	0.00	0.00	0.00	0.00
1	1.00	0.50	0.17	0.04	0.01	0.00
2	1.00	0.83	0.50	0.22	0.08	0.02
3	1.00	0.96	0.78	0.50	0.26	0.11
4	1.00	0.99	0.92	0.74	0.50	0.29
5	1.00	1.00	0.98	0.89	0.72	0.50

Lanchester P(A) is zero above the dashed lines, one below the dashed lines, and is zero in the dashed boxes.

(b)

BROWN (1955) p.29

SQUARE LAW

$a_f = b_f = 0$

$\alpha = 2\beta$

P(A)

a_0 \ b_0	0	1	2	3	4	5
0	-	0.00	0.00	0.00	0.00	0.00
1	1.00	0.67	0.33	0.13	0.04	0.01
2	1.00	0.93	0.73	0.48	0.26	0.12
3	1.00	0.99	0.93	0.79	0.57	0.37
4	1.00	1.00	0.98	0.93	0.81	0.64
5	1.00	1.00	1.00	0.98	0.93	0.83

Lanchester P(A) is zero above the dashed lines, one below the dashed lines, and is zero in the dashed box.

(c)

BROWN (1955) p.29

SQUARE LAW

$a_f = b_f = 0$

$\alpha = 9\beta$

P(A)

a_0 \ b_0	0	1	2	3	4	5
0	-	0.00	0.00	0.00	0.00	0.00
1	1.00	0.90	0.74	0.41	0.29	0.18
2	1.00	0.99	0.97	0.89	0.78	0.65
3	1.00	1.00	1.00	0.99	0.96	0.91
4	1.00	1.00	1.00	1.00	1.00	0.98
5	1.00	1.00	1.00	1.00	1.00	1.00

Lanchester P(A) is zero above the dashed lines, and one below the dashed lines.

(d)

[illegible] (1965) p.54

SQUARE LAW

$a_f = b_f = 0$

$\alpha = \beta$

P(A)

a_0 \ b_0	1	2	3	4	5	6
1	.500	.167	.042	.008	.001	.0002
2	.833	.500	.225	.081	.024	.006
3	.958	.775	.500	.260	.113	.042
4	.992	.919	.740	.500	.285	.139
5	.999	.975	.887	.715	.500	.303
6	.9998	.994	.958	.861	.697	.500

N.B.: P(A) = 0 above heavy line for L model.
= 1 below heavy line for L model.

Table IV-1

LEE (1979) p.10

SQUARE LAW

$a_f = b_f = 0$

α/β	a_o	b_o	SL P(A)	L P(A)
1.0	1.0	7.0	.0000248	0
1.0	2.0	6.0	.0081507	0
1.0	3.0	5.0	.1126240	0
1.0	4.0	4.0	.5000000	0
1.0	5.0	3.0	.8873759	1
1.0	6.0	2.0	.9938492	1
1.0	7.0	1.0	.9999751	1
2.0	1.0	7.0	.0007054	0
2.0	2.0	6.0	.0003894	0
2.0	3.0	5.0	.0187682	0
2.0	4.0	4.0	.8083533	1
2.0	5.0	3.0	.9812317	1
2.0	6.0	2.0	.9996105	1
2.0	7.0	1.0	.9992945	1
3.0	1.0	7.0	.0036160	0
3.0	2.0	6.0	.0000602	0
3.0	3.0	5.0	.0050362	1
3.0	4.0	4.0	.9146103	1
3.0	5.0	3.0	.9949637	1
3.0	6.0	2.0	.9999397	1
3.0	7.0	1.0	.9963839	1

(a)

KARR (1976) p.22

SQUARE LAW

$a_f = b_f = 0$

	P(A)			
	SL			L
β/α	a = 50; b = 50	a = 200; b = 200	a = 500; b = 500	All Cases
.80	.77	.91	1	1
.90	.74	.79	.88	1
.95	.54	.68	.73	1
1.05	.42	.34	.26	0
1.10	.23	.25	.11	0
1.20	.20	.08	0	0

(b)

Table IV-2

LEE AND WANNASILPA (1972) p.25

SQUARE LAW

$a_f = b_f = 0$

INITIAL FORCES		P(B)		
a_o	b_o	$\beta = .05,\ \alpha = .05$	$\beta = .1,\ \alpha = .3$	$\beta = .05,\ \alpha = .2$
1	3	0.9583	-	-
2	6	0.9938	0.87179	0.64338
3	9	0.9990	0.92326	0.84266
4	12	0.9998	0.95268	0.88234
5	15	0.9999	0.97028	0.91067
6	18	1.0	0.98110	0.93148
7	21	1.0	0.98787	0.94703
8	24	1.0	0.99219	0.95881
9	27	1.0	0.99554	0.96804
10	30	1.0	.	0.98188
.	.	.	.	.
.	.	.	.	.
20	60	1.0	.	.
L for P(B)		1.0	1.0	1.0

(a)

GYE AND LEWIS (1976) p.114

SQUARE LAW

$a_f = b_f = 0$

$\frac{\alpha}{\beta} = \frac{1}{25}$

a_o	P(B)				
	$b_o = 1$	2	3	4	5
1	0.962	0.999	1.000	1.000	
2	0.890	0.995	1.000	1.000	
3	0.795	0.984	0.999	1.000	
4	0.685	0.962	0.997	1.000	
5	0.571	0.926	0.993	1.000	
6	0.461	0.876	0.984	0.999	
7	0.360	0.813	0.970	0.997	
8	0.233	0.738	0.947	0.993	
9	0.200	0.656	0.916	0.987	
10	0.143	0.571	0.875	0.977	
11	0.099	0.486	0.826	0.962	
12	0.067	0.405	0.768	0.941	
13	0.044	0.330	0.703	0.914	
14	0.028	•0.264	0.634	0.879	
15	0.018	0.207	0.563	0.838	
16	0.011	0.160	0.492	0.790	
17	0.006	0.121	0.423	0.737	
18	0.004	0.090	0.359	0.679	
19	0.002	0.066	0.300	0.619	
20	0.001	0.047	0.246	0.557	
21	0.001	0.033	0.200	0.495	
22	0.000	0.023	0.160	0.434	
.					
.					
.					
26	-	-	-	-	0.496

NOTE. Above the heavy lines (——), for L, P(B) = 1.0; and below the lines, P(B) for L = 0.

(b)

Table IV-3

KARR (1975a) p.21

SQUARE LAW

$p_A = p_B = 0.5$
$a_f = .67a_o,\ b_f = .79b_o$
P(B) Computed on 50 Replications

a_o	b_o	r_A	r_B	$\frac{p_B r_B b_o^2}{p_A r_A a_o^2}$	P(B)
50	50	1	1.0	1.0	0.16
50	50	1	1.2	1.2	0.25
50	50	1	1.4	1.4	0.45
50	50	1	1.6	1.6	0.58
50	50	1	1.8	1.8	0.81
50	75	1	0.1	0.225	0
50	75	1	0.2	0.45	0
50	75	1	0.3	0.675	0.01
50	75	1	0.4	0.9	0.05
50	75	1	0.5	1.125	0.2
50	75	1	0.6	1.35	0.48
50	75	1	0.7	1.575	0.58
50	75	1	0.8	1.8	0.69
50	75	1	0.9	2.025	0.85
50	100	1	0.1	0.4	0
50	100	1	0.2	0.8	0.02
50	100	1	0.3	1.2	0.29
50	100	1	0.4	1.6	0.6[illegible]
50	100	1	0.5	2.0	0.8
50	100	1	0.6	2.4	0.9
50	100	1	0.7	2.8	0.98
50	100	1	0.8	[illegible]	0.99
50	150	1	0.1	[illegible]	0.03
50	150	1	0.11	0.99	0.05
50	150	1	0.12	1.08	0.12
50	150	1	0.13	1.17	0.19
50	150	1	0.14	1.26	0.34
50	150	1	0.15	1.35	0.39
50	150	1	0.16	1.44	0.45
50	150	1	0.17	1.53	0.55
50	150	1	0.18	1.62	0.62
50	150	1	0.19	1.71	0.61
50	150	1	0.2	1.8	0.71
100	100	1	1.0	1.0	[illegible]
100	100	1	1.1	1.1	0.19
100	100	1	1.2	1.2	[illegible]
100	100	1	1.3	1.3	[illegible]
100	100	1	1.4	1.4	0.1
100	100	1	1.5	1.5	0.55
100	100	1	1.6	1.6	0.62
100	100	1	1.7	1.7	0.72
100	100	1	1.8	1.8	0.73
100	100	1	1.9	1.9	0.93

For L, P(B) = 0 above solid lines and = 1 below.

Table IV-4

KARR (1975a) p.17

SQUARE LAW

$P_A = P_B = 0.5$

$a_f = 0.5\ a_o,\ b_f = 0.5\ b_o$

P(B) Computed on 50 Replications

b_o	a_o	r_B	r_A	$\frac{P_B r_B b_o^2}{P_A r_A a_o^2}$	P(B)
50	50	1	1	1	0.56
50	50	2	1	2	0.98
50	50	1.5	1	1.5	0.94
50	50	1.25	1	1.25	0.8
50	50	1.1	1	1.1	0.74
50	50	1.05	1	1.05	0.54
50	50	3	1	3	1
50	100	1	1	0.25	0
50	100	2	1	0.5	0
50	100	3	1	0.75	0.12
50	100	3.5	1	0.875	0.38
50	100	4	1	1	0.54
50	100	4.5	1	1.125	0.63
50	100	5	1	1.25	0.88
50	100	5.5	1	1.375	0.99
50	100	7	1	1.75	1
50	150	1	1	0.111	0
50	150	2	1	0.222	0
50	150	3	1	0.333	0
50	150	4	1	0.444	0
50	150	5	1	0.555	0
50	150	6	1	0.666	0.06
50	150	7	1	0.777	0.14
50	150	8	1	0.888	0.3
50	150	9	1	1	0.5
50	150	10	1	1.111	0.7
50	150	11	1	1.222	0.84
50	150	12	1	1.333	0.92
50	150	13	1	1.441	0.92

b_o	a_o	r_B	r_A	$\frac{P_B r_B b_o^2}{P_A r_A a_o^2}$	P(B)
100	150	1	1	0.445	0
100	150	2	1	0.89	0.26
100	150	2.25	1	1	0.46
100	150	2.5	1	1.11	0.72
100	150	3	1	1.33	0.94
100	150	4	1	1.78	1
200	200	0.75	1	0.75	0
200	200	0.80	1	0.80	0.08
200	200	0.85	1	0.85	0.13
200	200	0.90	1	0.90	0.25
200	200	0.95	1	0.95	0.31
200	200	1.00	1	1.00	0.51
200	200	1.05	1	1.05	0.68
200	200	1.10	1	1.10	0.79
200	200	1.15	1	1.15	0.8
200	200	1.20	1	1.20	0.87
200	200	1.25	1	1.25	0.91
500	500	0.75	1	0.75	0
500	500	0.80	1	0.80	0
500	500	0.85	1	0.85	0.04
500	500	0.90	1	0.90	0.11
500	500	0.95	1	0.95	0.26
500	500	1.00	1	1.00	0.53
500	500	1.05	1	1.05	0.73
500	500	1.10	1	1.10	0.88
500	500	1.15	1	1.15	0.96

For 5, P(B) = 0 above solid lines and = 1 below.

(a)

DALY (1984) p 12

SQUARE LAW

$a_f = b_f = 0$

CASE #	a_o	b_o	α/β	L P(A)	SL P(A)
1	2	3	2.25	0 (L Parity)	.523
2	1	Very Large	b_o^2	0 (L Parity)	.61
3	9	4	1/5	> 1/2	.472
4	14	4	1/12	< 1/2	.464

(b)

Table IV-5

FARRELL (1976) p.28

SQUARE LAW

$a_0 = 12,\ b_0 = 7$

$a_f = b_f = 0$

$\alpha = .003,\ \beta = .01$

(a)

FOR L, P(A) = 0, P(B) = 1				
	P(B,t)		P(A,t)	
t	SIMULATION	NORMAL (Approximation)	SIMULATION	NORMAL (Approximation)
80	.01	.00	.01	.00
160	.17	.14	.14	.08
240	.40	.38	.26	.23
280	.47	.46	.31	.29
400	.58	.56	.36	.37

Simulation - 5000 replications.

Approximations - Farrell and Freedman (1975).

FARRELL (1976) p.28

SQUARE LAW

$a_0 = b_0 = 10$

$a_f = b_f = 0$

$\alpha = .003,\ \beta = .01$

(b)

FOR L, P(A) = 0, P(B) = 1				
	P(B,t)		P(A,t)	
t	SIMULATION	NORMAL (Approximation)	SIMULATION	NORMAL (Approximation)
30	.00	.00	.00	.00
60	.06	.04	.00	.00
90	.30	.27	.00	.00
120	.59	.57	.00	.00
150	.79	.76	.00	.00

Simulation - 5000 replications.

Approximations - Farrell and Freedman (1975).

FARRELL (1976) p.28

SQUARE LAW

$a_0 = 2,\ b_0 = 4$

$a_f = b_f = 0$

$\alpha = .02,\ \beta = .01$

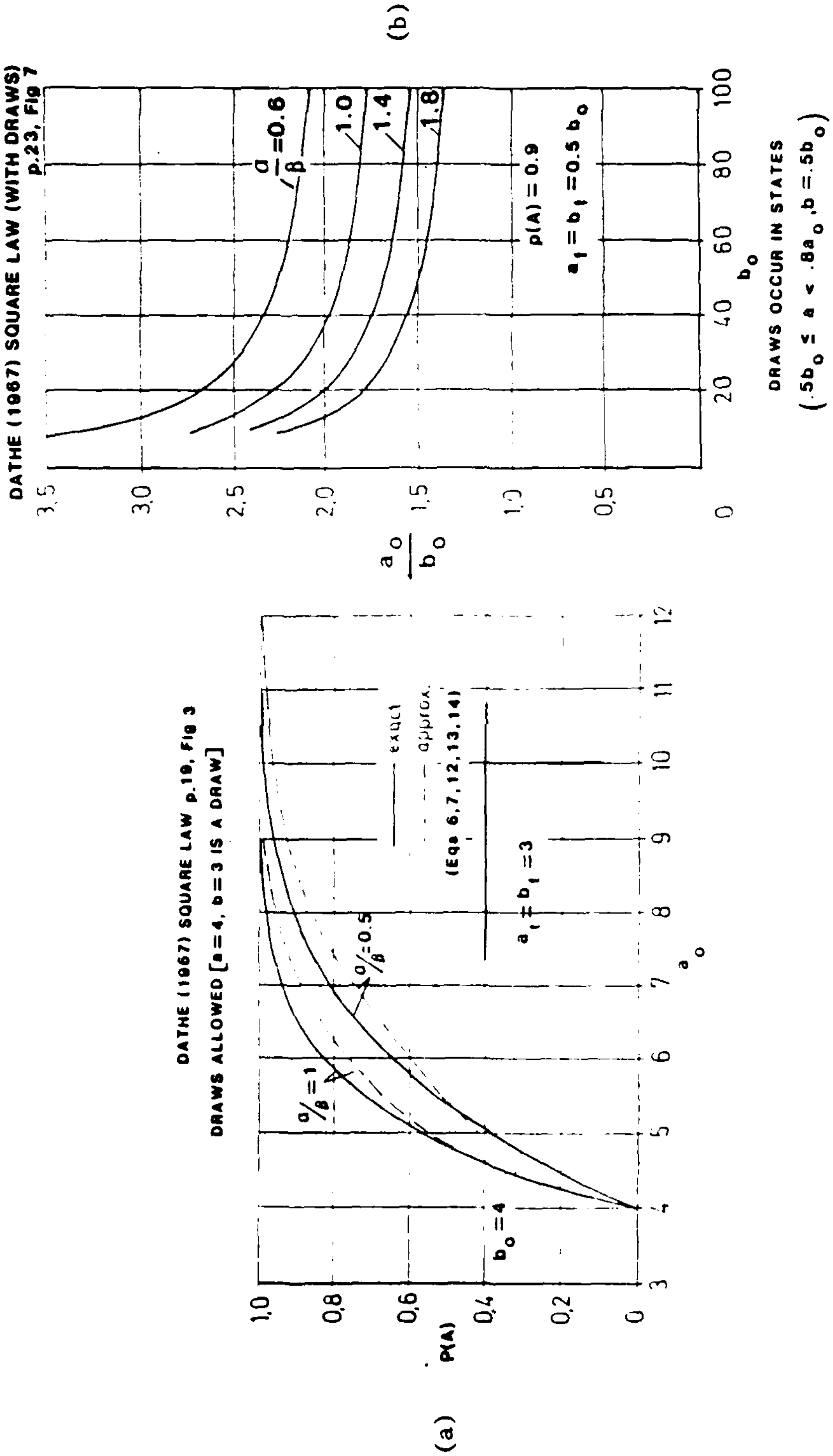
(c)

FOR L, P(A) = 0, P(B) = 1				
	P(B,t)		P(A,t)	
t	SIMULATION	NORMAL (Approximation)	SIMULATION	NORMAL (Approximation)
20	.17	.07	.01	.00
40	.38	.31	.04	.01
60	.53	.48	.10	.05
80	.63	.59	.15	.10
100	.68	.66	.19	.15

Simulation - 5000 replications.

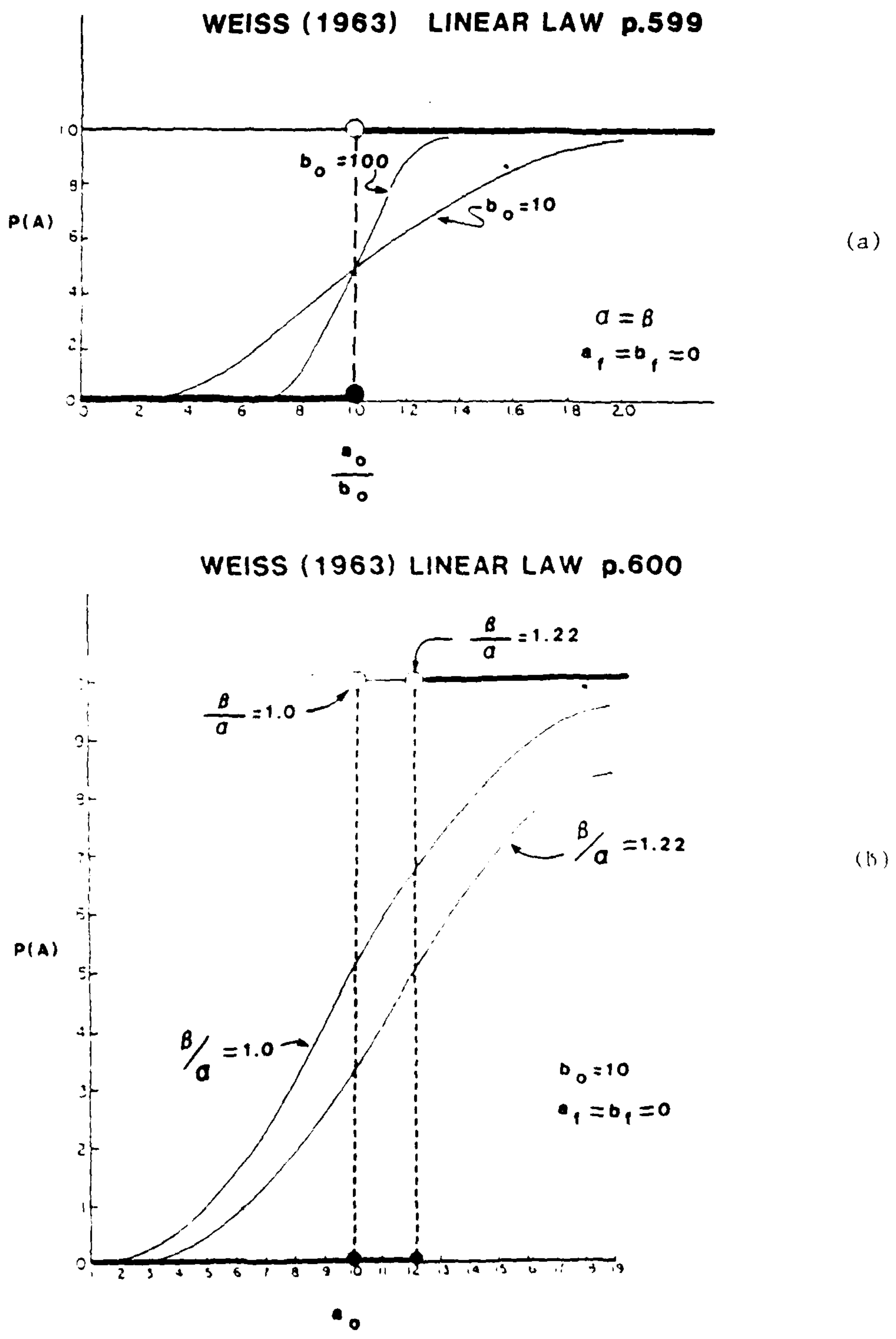
Approximations - Farrell and Freedman (1975).

Table IV-6



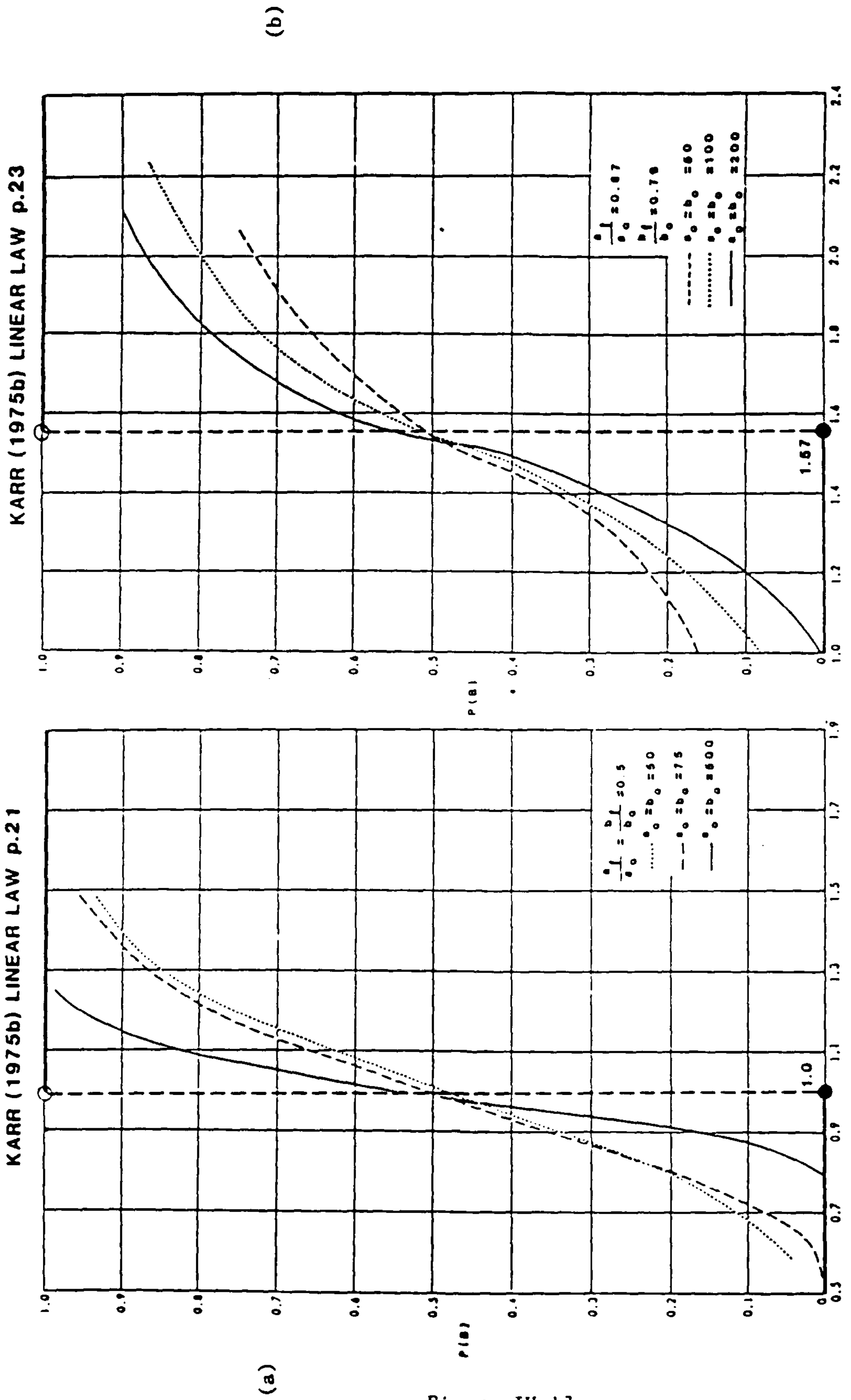
DATHE (1967) SQUARE LAW (WITH DRAWS) p.23, Fig 7
$\frac{a_o}{b_o}$
b_o
$\alpha/\beta = 0.6$
1.0
1.4
1.8
p(A) = 0.9
$a_1 = b_1 = 0.5\ b_o$
DRAWS OCCUR IN STATES
$(.5b_o \le a < .8a_o, b = .5b_o)$
(b)
DATHE (1967) SQUARE LAW p.19, Fig 3
DRAWS ALLOWED [a = 4, b = 3 IS A DRAW]
P(A)
a_o
$b_o = 4$
$\alpha/\beta = 1$
$\alpha/\beta = 0.5$
exact
approx.
(Eqs 6,7,12,13,14)
$a_1 = b_1 = 3$
(a)

Figure IV-15



WEISS (1963) LINEAR LAW p.599
P(A)
b o =100
b o =10
α = β
a f = b f = 0
a o
b o
(a)
WEISS (1963) LINEAR LAW p.600
β/α =1.22
β/α =1.0
P(A)
b o =10
a f = b f = 0
a o
(b)

Figure IV-16



KARR (1975b) LINEAR LAW p.21
KARR (1975b) LINEAR LAW p.23
(a)
(b)
1.0
1.57
P(B)

Figure IV-17

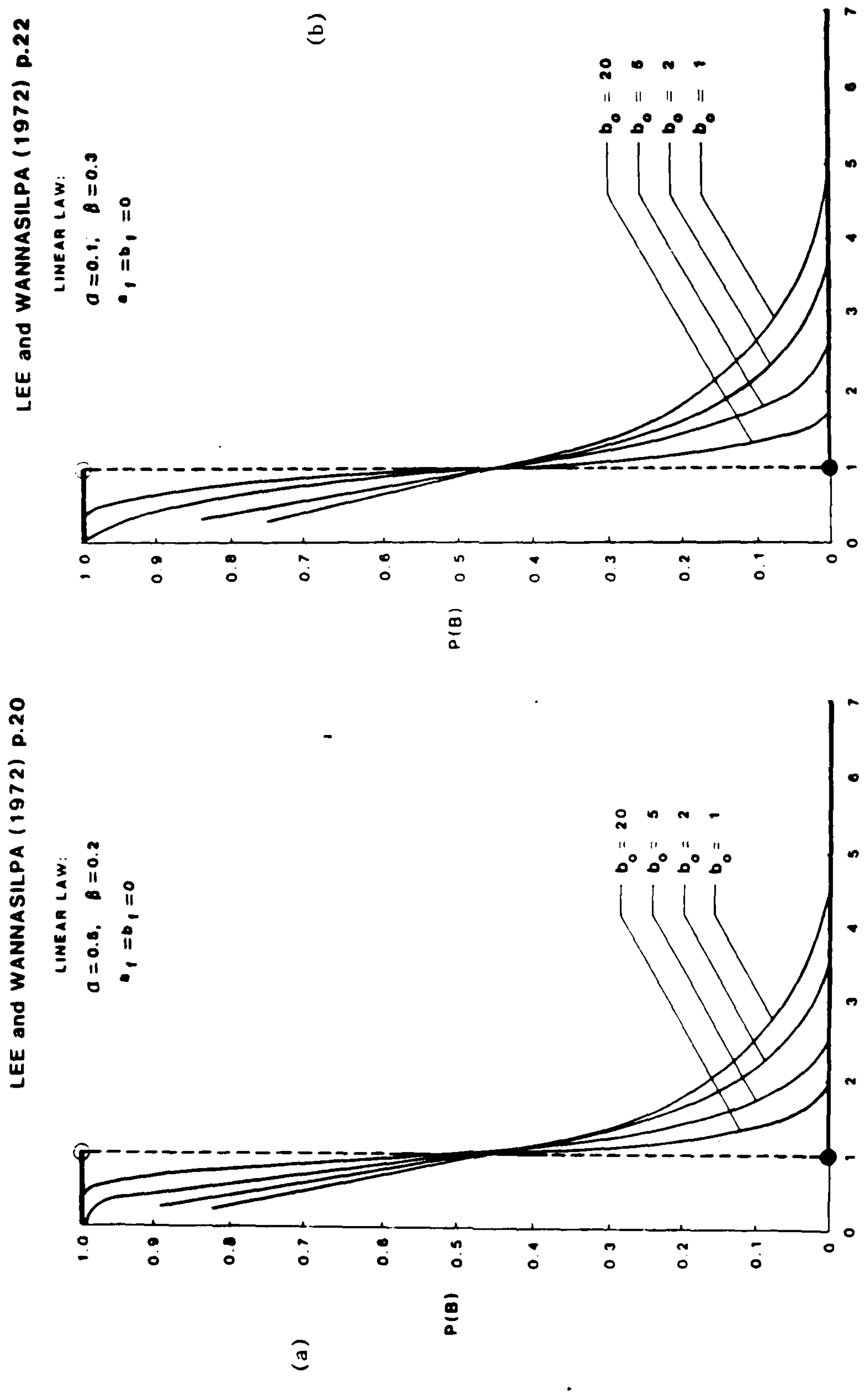

LEE and WANNASILPA (1972) p.20
LINEAR LAW:
α = 0.5, β = 0.2
a₁ = b₁ = 0
b₀ = 20
b₀ = 5
b₀ = 2
b₀ = 1
P(B)
(a)
LEE and WANNASILPA (1972) p.22
LINEAR LAW:
α = 0.1; β = 0.3
a₁ = b₁ = 0
b₀ = 20
b₀ = 5
b₀ = 2
b₀ = 1
P(B)
(b)

Figure IV-18

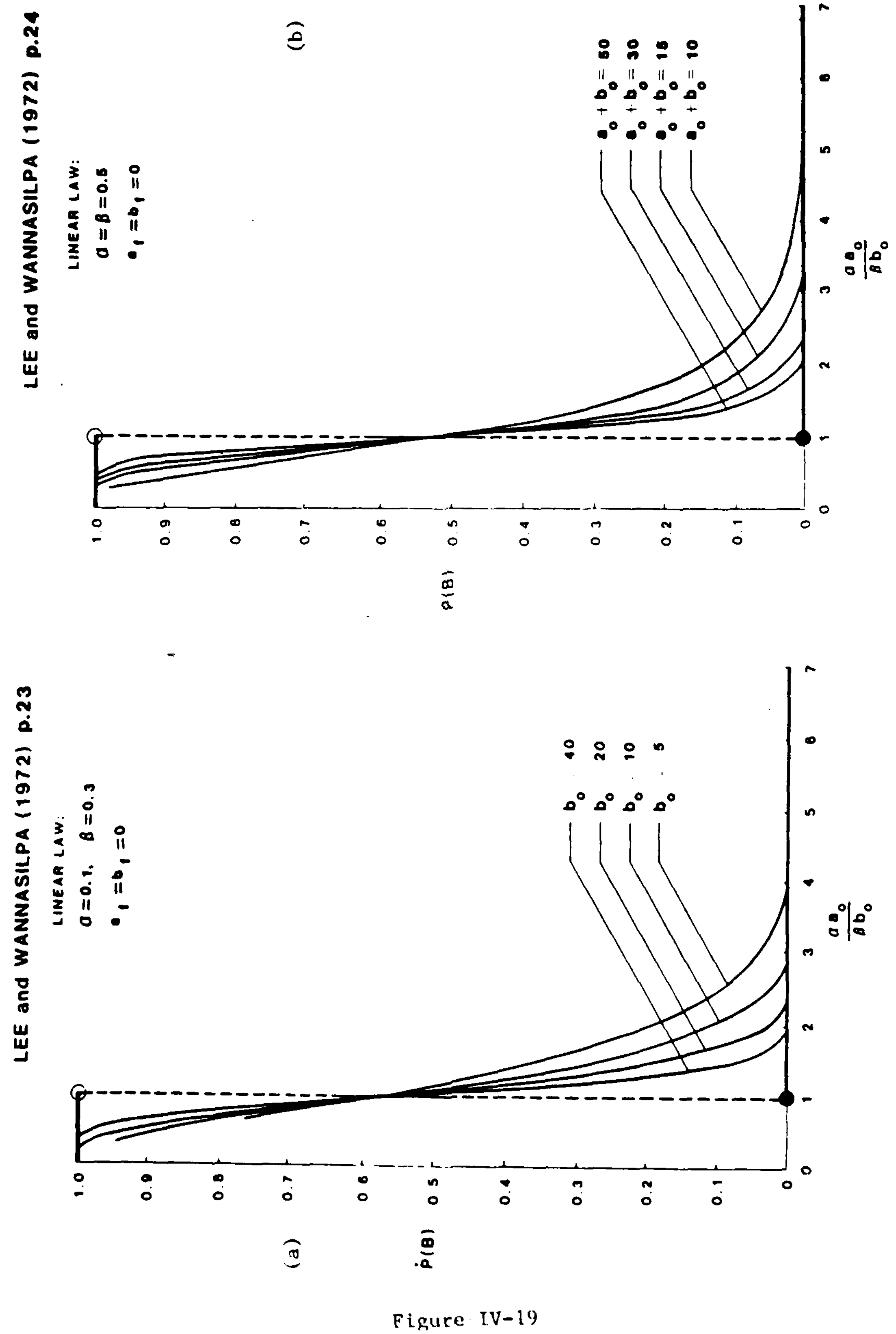

(a)
LEE and WANNASILPA (1972) p.23
LINEAR LAW:
α=0.1, β=0.3
a_f = b_f = 0
b_o 40
b_o 20
b_o 10
b_o 5
P(B)
αa_o / βb_o
(b)
LEE and WANNASILPA (1972) p.24
LINEAR LAW:
α = β = 0.5
a_f = b_f = 0
a_o + b_o = 50
a_o + b_o = 30
a_o + b_o = 15
a_o + b_o = 10
P(B)
αa_o / βb_o

Figure IV-19

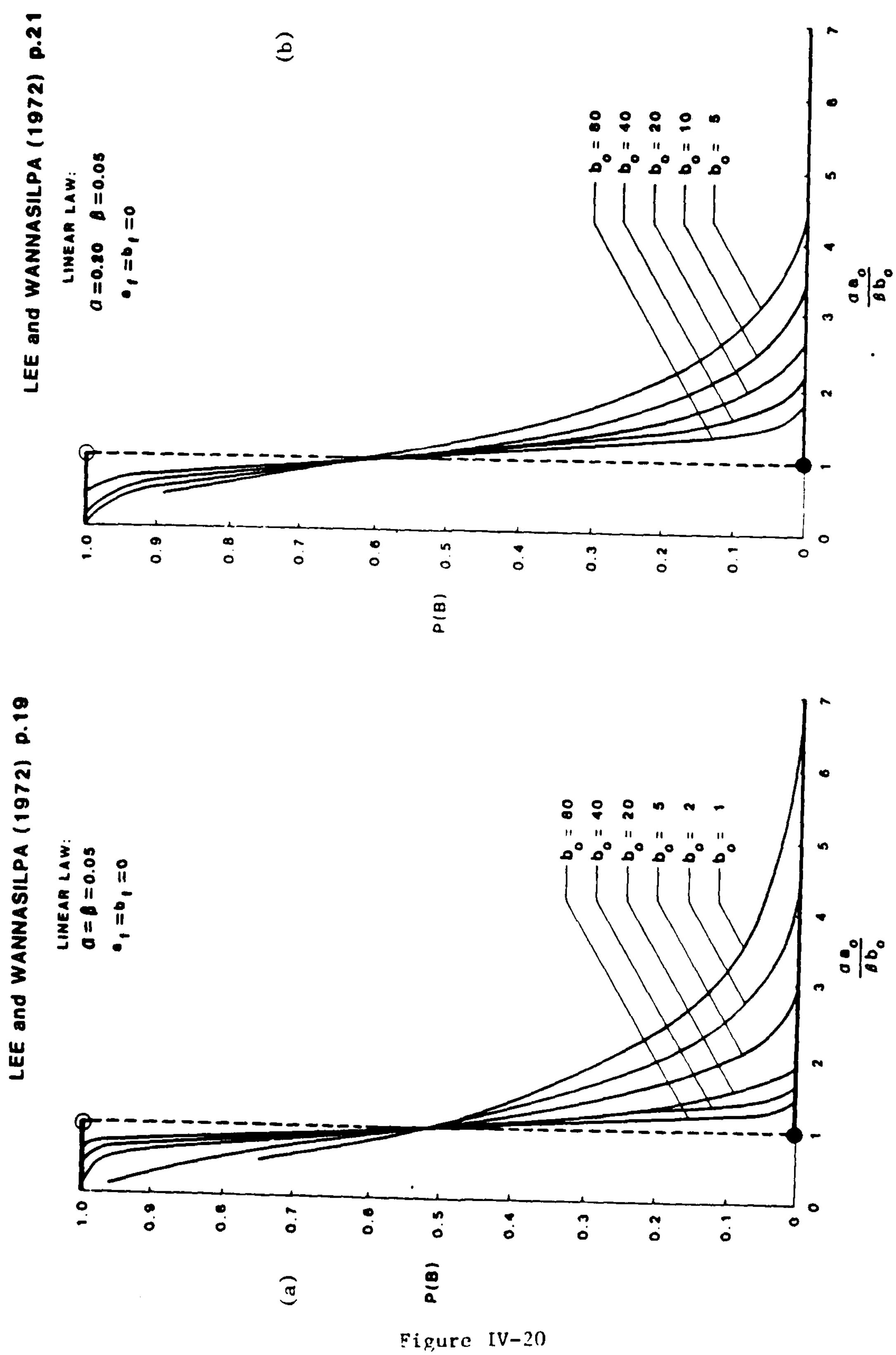

LEE and WANNASILPA (1972) p.19
LINEAR LAW:
α = β = 0.05
a_f = b_f = 0
b_o = 80
b_o = 40
b_o = 20
b_o = 5
b_o = 2
b_o = 1
P(B)
αa_o / βb_o
(a)
LEE and WANNASILPA (1972) p.21
LINEAR LAW:
α = 0.20 β = 0.05
a_f = b_f = 0
b_o = 80
b_o = 40
b_o = 20
b_o = 10
b_o = 5
P(B)
αa_o / βb_o
(b)

Figure IV-20

KARR (1975b) p.20

LINEAR LAW

$$\frac{a_f}{a_o} = \frac{b_f}{b_o} = 0.5$$

$$f = \frac{p_B r_B b_o}{p_A r_A a_o}$$

$a_o = 50,\ b_o = 75$		$a_o = 50,\ b_o = 100$		$a_o = b_o = 200$	
f	P(B)	f	P(B)	f	P(B)
0.240	0.00	0.4	0.00	0.75	0.01
0.489	0.00	0.5	0.00	0.80	0.05
0.738	0.15	0.6	0.03	0.85	0.16
0.987	0.48	0.7	0.09	0.90	0.22
1.236	0.81	0.8	0.23	0.95	0.38
1.485	0.97	0.9	0.36	1.00	0.47
1.734	0.99	1.0	0.54	1.05	0.58
1.983	1.00	1.1	0.69	1.10	0.76
-	-	1.2	0.76	1.15	0.81
-	-	1.3	0.83	1.20	0.92
-	-	1.4	0.92	1.25	0.93
-	-	1.5	0.94	-	-
-	-	1.6	0.97	-	-
-	-	1.7	0.99	-	-

N.B.: L P(B) = 0 above dashed horizontal lines.
= 1 below dashed horizontal lines.

(a)

KARR (1975b) p.22

LINEAR LAW

$$\frac{a_f}{a_o} = .67,\ \frac{b_f}{b_o} = .79$$

$$f = \frac{p_B r_B b_o}{p_A r_A a_o}$$

$a_o = 50,\ b_o = 75$	
f	P(B)
0.75	0.00
0.90	0.04
1.05	0.11
1.20	0.22
1.35	0.24
1.50	0.42
1.65	0.58
1.80	0.67
1.95	0.76
2.10	0.71

N.B.:

L P(B) = 0 above dashed horizontal lines.
= 1 below dashed horizontal lines.

(b)

Table IV-7

LEE AND WANNASILPA (1972) p.25

LINEAR LAW

$a_f = b_f = 0$

INITIAL FORCES		P(B)		
a_0	b_0	$\beta = .05,\ \alpha = .05$	$\beta = .1,\ \alpha = .3$	$\beta = .05,\ \alpha = .2$
1	3	0.8750	0.57812	0.51200
2	6	0.9375	0.55505	0.57680
3	9	0.9672	0.54480	0.61740
4	12	0.9824	0.53871	0.64816
5	15	0.9904	0.53458	0.67329
6	18	0.9947	0.53158	0.69469
7	21	0.9970	0.52917	0.71339
8	24	0.9983	0.52727	0.73003
9	27	0.9991	0.52570	0.74501
10	30	0.9995	0.52437	0.75864
.	.	.	.	.
.	.	.	.	.
20	60	1.0	0.51720	0.85079
L for P(B)		1.0	0	0

Table IV-8

LEE and WANNASILPA (1968) p.33

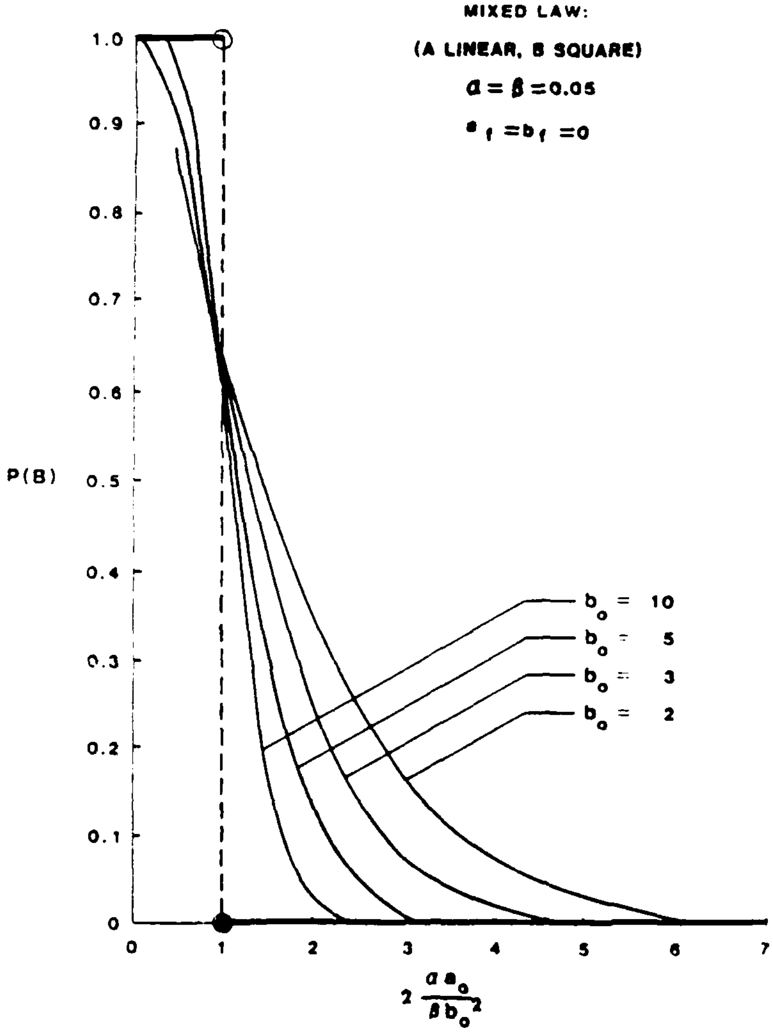


Figure IV-21

LEE AND WANNASILPA (1972) p.25

MIXED LAW

$a_f = b_f = 0$

INITIAL FORCES		P(B)
a_0	b_0	$\beta = .05,\ \alpha = .05$
1	3	-
2	6	0.1600
3	9	0.1348
4	12	0.3362
5	15	0.4536
6	18	0.5742
7	21	0.6874
8	24	0.7833
9	27	0.8582
10	30	0.9123
.	.	.
.	.	.
20	60	1.0
L for P(B)		1.0

Table IV-9

SPRINGALL (1968) MODEL p.156

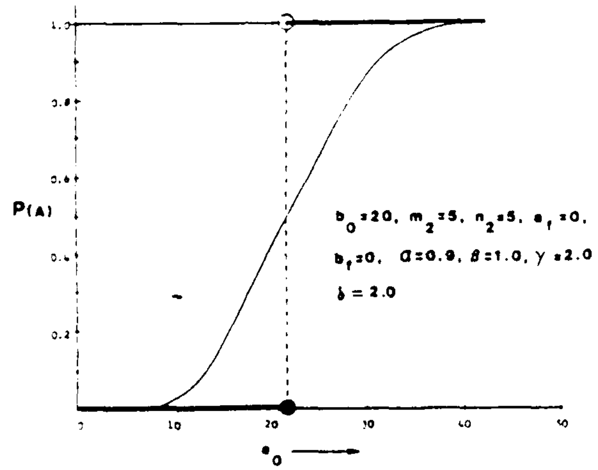


Figure IV-22

SPRINGALL (1968) pp.168,169

SPRINGALL MODEL

$m_2 = 5,\ \alpha = 0.9,\ \beta = 1.0,\ \delta = \gamma = 2.0,\ a_f = b_f = 0$

(a)

a_0 AND b_0	P(B)
5	0.53560
10	0.55793
20	0.58710
30	0.60847
40	0.62600
50	0.64112
60	0.65453

$m_2 = 5,\ \alpha = 0.9,\ \beta = 3.0,\ \delta = \gamma = 2.0,\ a_f = b_f = 0$

(b)

a_0 AND b_0	P(B)
5	0.88567
10	0.97080
20	0.99763
30	0.99979
40	0.99998

SPRINGALL (1968) p. [illegible]

SPRINGALL MODEL

$a_0 = 60,\ b_0 = 60,\ \alpha = 0.9,\ \beta = 1.0,\ \gamma = 2.0,\ \delta = 2.0$

$a_f = b_f = 5$

(c)

m_2 and n_2	P(A)
10	0.32519
15	0.31638
20	0.31211
30	0.30832
40	0.30690
50	0.30637
60	0.30624

Table IV-10

V. THE VARIANCE OF THE PROCESS

In this section standard deviations are shown directly in Figures V-1 through V-9 and Tables V-1 through V-6. The remainder of the section shows state probability density functions and probability mass functions in various forms. The pdf's and pmf's are given here even though they do not explicitly display variance (or standard deviation) because they visually illustrate that dispersion is usually very large.

Other sections contain distribution information which should also be examined when variance is considered. These other tables and figures are;

(1) The Square Law; Figure II-1 (gives 5th and 95th percentiles), II-2, II-3, (all the preceding are terminal distributions), III-1, III-2 (the preceding two are distributions functions of time-duration, III-3 (pdf or time-duration), VI-3 through VI-12, and Tables II-1, II-2(a),(b), (preceding two are terminal), III-1(a), (c) (time-duration variances), VI-3, VI-4 and VI-5 (the last three are joint distributions at various times).

(2) The Square Law with draws; Table III-2 (pdf and df's of time-duration).

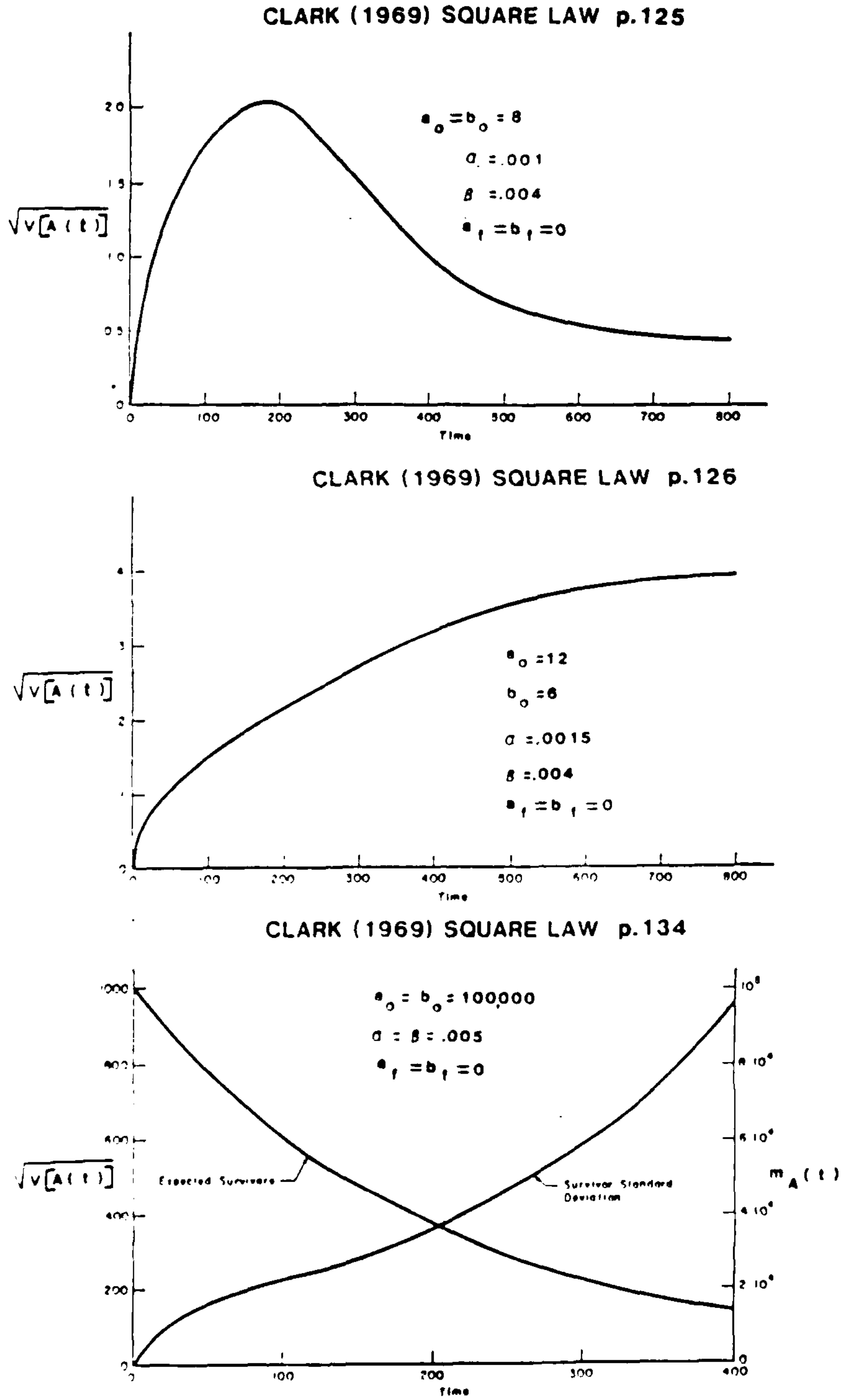
(3) The Linear Law; Table II-3 (terminal).

(4) The Weale Special Model; Table III-3(a) (time-duration), VI-6, and VI-7 (last two are joint distributions).

Particular care must be used in interpreting the figures on state distributions which are discrete as opposed to time-duration distributions which are continuous. That is, they are all probability mass functions even though they are variously depicted as histograms or as continous. Only figure V-23 is correctly portrayed. The histograms should be read as their height at mid-points of the rectangles and the continuous curves only at integers on the abscissa.

Also, the reader should be aware of the peculier manner in which Figures V-12, V-13, and V-14 have been prepared. These are frequency diagrams (not histograms) and are to be read as having values only at the right hand edge of each rectangle and whose magnitude is the rectangle height on the left. However the x(t) and y(t) values (the vertical dashed lines) are located at the middle of the corresponding rectangle. For example, in Figure V-12(a), x(t) = 12 is located at 11.5, x(t) = 3 at 2.5 and so forth.

The examples shown in this section amply support the contention that variance is indeed very important.



(a)

(b)

(c)

Figure V-1

FARRELL (1976) p.27

SQUARE LAW

$a_0 = b_0 = 10$

$a_f = b_f = 0$

$\alpha = .003,\ \beta = .01$

t	$\mu = m_A(t);\ \sigma = \sqrt{V[A(t)]}$						$\mu = m_B(t);\ \sigma = \sqrt{V[B(t)]}$					
	μ_1	μ_2	μ_3	σ_1	σ_2	σ_3	μ_1	μ_2	μ_3	σ_1	σ_2	σ_3
30	7.12	7.13	7.12	1.72	1.68	1.72	9.23	9.20	9.23	0.89	0.91	0.89
60	4.44	4.54	4.47	2.44	2.32	2.37	8.71	8.70	8.70	1.19	1.21	1.17
90	1.87	2.47	2.38	3.08	2.28	2.38	8.43	8.40	8.32	1.40	1.40	1.28
120	-0.64	1.18	1.18	3.71	1.80	1.94	8.37	8.26	8.15	1.60	1.54	1.36
150	-3.18	0.55	0.60	4.39	1.30	1.50	8.55	8.18	8.15	1.92	1.64	1.44

SUBSCRIPTS:

1 - Snow (1948) approximation. (Zero absorption probabilities.)
2 - Simulation (5000 replications).
3 - Farrell & Freedman (1975) approximations.

(a)

FARRELL (1976) p.27

SQUARE LAW

$a_0 = 2,\ b_0 = 4$

$a_f = b_f = 0$

$\alpha = .02,\ \beta = .01$

t	$\mu = m_A(t);\ \sigma = \sqrt{V[A(t)]}$						$\mu = m_B(t);\ \sigma = \sqrt{V[B(t)]}$					
	μ_1	μ_2	μ_3	σ_1	σ_2	σ_3	μ_1	μ_2	μ_3	σ_1	σ_2	σ_3
20	1.27	1.32	1.30	0.87	0.74	0.82	3.35	3.36	3.23	0.84	0.83	0.82
40	0.64	0.90	0.98	1.26	0.91	0.93	2.97	2.94	2.84	1.23	1.13	1.07
60	0.07	0.68	0.65	1.66	0.80	0.89	2.93	2.67	2.53	1.69	1.32	1.24
80	-0.51	0.55	0.52	2.15	0.77	0.81	2.92	2.54	2.37	2.31	1.42	1.37
100	-1.12	0.48	0.45	2.78	0.76	0.76	3.24	2.45	2.28	3.16	1.49	1.45

SUBSCRIPTS:

1 - Snow (1948) approximation. (Zero absorption probabilities.)
2 - Simulation (5000 replications).
3 - Farrell & Freedman (1975) approximations.

(b)

FARRELL (1976) p.27

SQUARE LAW

$a_0 = 12,\ b_0 = 7$

$a_f = b_f = 0$

$\alpha = .003,\ \beta = .01$

t	$\mu = m_A(t);\ \sigma = \sqrt{V[A(t)]}$						$\mu = m_B(t);\ \sigma = \sqrt{V[B(t)]}$					
	μ_1	μ_2	μ_3	σ_1	σ_2	σ_3	μ_1	μ_2	μ_3	σ_1	σ_2	σ_3
80	7.39	7.40	7.39	2.35	2.33	2.35	4.71	4.69	4.71	1.60	1.59	1.60
160	4.22	4.54	4.43	3.90	3.26	3.31	3.34	3.43	3.33	2.43	2.10	2.08
240	1.38	3.30	3.22	6.18	3.54	3.62	2.62	2.91	2.75	3.58	2.32	2.32
280	0.86	3.04	3.01	7.75	3.59	3.72	2.45	2.78	2.64	4.38	2.39	2.40
400	-2.10	2.65	2.87	15.09	3.60	3.88	2.67	2.72	2.52	8.27	2.46	2.51

SUBSCRIPTS:

1 - Snow (1948) approximation. (Zero absorption probabilities.)
2 - Simulation (5000 replications).
3 - Farrell & Freedman (1975) approximations.

(c)

Table V-I

PERLA AND LEHOCZKY (1977) p.27

<u>SQUARE LAW</u>

$a_\varphi = b_\varphi = 0$

$\alpha = \beta = .05$

$t = 15$

(a_φ, b_φ)	$\sqrt{V[A(t)]}$ S	D	|S−D|/S	$\sqrt{V[B(t)]}$ S	D	|S−D|/S
(20,20)	1.87 (.053)	1.98	.0284	3.94 (.026)	1.98	.0102
(25,25)	4.42 (.050)	4.45	.0068	4.39 (.064)	4.45	.0137
(30,30)	4.84 (.004)	4.88	.0083	4.83 (.091)	4.88	.0104
(40,40)	5.56 (.136)	5.63	.0053	5.64 (.027)	5.63	.0018
(50,50)	6.27 (.031)	6.30	.0048	6.31 (.033)	6.30	.0016

S = Simulation, 6000 replications. Numbers in parentheses are standard deviations of the S estimates.

D = Perla & Lehoczky (1977) diffusion approximation.

(a)

PERLA AND LEHOCZKY (1977) p.28

<u>SQUARE LAW</u>

$a_\varphi = b_\varphi = 0$

$\alpha = .075,\ \beta = .030$

$t = 10$

(a_φ, b_φ)	$\sqrt{V[A(t)]}$ S	D	|S−D|/S	$\sqrt{V[B(t)]}$ S	D	|S−D|/S
(50,20)	3.59 (.021)	3.68	.0251	3.58 (.035)	3.84	.0726
(75,30)	4.43 (.055)	4.51	.0181	4.61 (.074)	4.71	.0217
(100,40)	5.26 (.049)	5.21	.0095	5.40 (.096)	5.43	.0056
(125,50)	5.80 (.070)	5.82	.0034	6.08 (.088)	6.08	.0000
(250,100)	8.25 (.238)	8.24	.0012	8.62 (.054)	8.59	.0035

S = Simulation, 6000 replications. Numbers in parentheses are standard deviations of the S estimates.

D = Perla & Lehoczky (1977) diffusion approximation.

(b)

CLARK (1969) p.127

<u>SQUARE LAW</u>

$a_\varphi = b_\varphi = 0$

a_φ	b_φ	α	β	t	$\sqrt{V[A(t)]}$	$\sqrt{V[B(t)]}$
6	6	.001	.004	1500.	.482	1.273
6	6	.0015	.004	1500.	.928	1.704
6	6	.002	.004	1500.	1.340	2.005
6	6	.004	.004	2500.	2.183	2.183
8	8	.001	.004	1500.	.390	1.434
8	8	.0015	.004	1500.	.908	2.000
8	8	.002	.004	1500.	1.460	2.444
8	8	.004	.004	2500.	2.740	2.740
12	6	.001	.004	2500.	3.934	2.136
12	6	.0015	.004	1500.	4.069	1.663
12	6	.002	.004	1500.	3.595	1.165
12	6	.004	.004	1500.	1.979	.274

(c)

Table V-2

WEALE (1972) pp.49,50

<u>SQUARE LAW</u>

$a_0 = b_0 = 10$

$a_f = b_f = 0$

$\alpha = 0.05, \ \beta = 0.025$

T	$m_A(t)$	$m_B(t)$	V[A(t)]	V[B(t)]	p*
0.0	10.000000	10.000000	0.000000	0.000000	1.00000
1.0	9.756199	9.506146	0.494256	0.244008	1.00000
2.0	9.524594	9.024177	0.979298	0.477044	1.00000
3.0	9.304896	8.553439	1.457676	0.700587	1.00000
4.0	9.096330	8.093493	1.932303	0.916068	1.00000
5.0	8.900137	7.643616	2.406421	1.124882	0.99999
6.0	8.714570	7.203294	2.882612	1.328389	0.99988
7.0	8.539897	6.771978	3.363666	1.527925	0.99890
8.0	8.375909	6.349127	3.852293	1.724810	0.99466
9.0	8.222374	5.934213	4.351239	1.920352	0.98252
10.0	8.079128	5.526718	4.863219	2.115856	0.95673
11.0	7.945981	5.126132	5.391392	2.312631	0.91259
12.0	7.822763	4.731954	[illegible]	2.511995	0.84826
13.0	7.709334	4.343692	6.505546	2.715285	0.76546
14.0	7.605538	3.960860	[illegible]	2.923861	0.66884
15.0	7.511250	3.582980	7.717751	3.139116	0.56469
16.0	7.426352	3.209579	8.368096	3.362480	0.45964
17.0	7.350737	2.840190	9.052149	3.595432	0.35954
18.0	7.284313	2.474352	9.773282	3.839505	0.26391
19.0	7.226994	2.111607	10.534967	4.096293	0.19073

p; probability mass **not** absorbed ≥ the entry.*

Table V-3

CLARK (1969) p.154

SQUARE LAW (WITH STOCHASTIC ACQUISITION)

$a_f = b_f = 0$

a_0	b_0	α	β	t	$\sqrt{V[A(t)]}$	$\sqrt{V[B(t)]}$
6	6	.001	.004	5000.	.284	1.325
6	6	.0015	.004	5000.	.605	1.662
6	6	.002	.004	5000.	.944	1.872
6	6	.004	.004	5000.	1.816	1.816
8	3	.001	.004	5000.	.202	1.528
8	8	.0015	.004	5000.	.542	1.983
3	8	.002	.004	5000.	.966	2.298
8	3	.004	.004	5000.	2.242	2.242
12	6	.001	.004	5000.	2.208	1.879
12	6	.0015	.004	5000.	3.276	1.743
12	6	.002	.004	5000.	3.662	1.389
12	6	.004	.004	5000.	2.866	.400

Acquisition Probabilities - ($p = q = .85$).

Table V-4

CLARK (1969) p.128

LINEAR LAW

$a_f = b_f = 0$

a_0	b_0	α	β	t	$\sqrt{V[A(t)]}$	$\sqrt{V[B(t)]}$
6	6	.001	.004	1500.	.241	1.346
6	6	.0015	.004	1500.	.530	1.642
6	6	.002	.004	1500.	.840	1.811
6	6	.004	.004	2500.	1.678	1.678
8	8	.001	.004	1500.	.158	1.571
8	8	.0015	.004	1500.	.441	1.954
8	8	.002	.004	1500.	.808	2.192
8	8	.004	.004	2500.	1.990	1.990
12	6	.001	.004	1500.	1.283	1.723
12	6	.0015	.004	1500.	2.271	1.743
12	6	.002	.004	2500.	2.987	1.493
12	6	.004	.004	1500.	3.120	.565

Table V-5

SPRINGALL (1968) p.158

SPRINGALL MODEL

$m_2 = 5,\ \alpha = 0.9,\ \beta = 3.0,\ \delta = 2.0,\ \gamma = 2.0$

$a_f = b_f = 0$

$b_0 = a_0$	P(B)	V[B(∞)]	V[A(∞)]
5	0.88567	2.436	0.430
10	0.97080	5.522	0.168
20	0.99763	11.173	0.018
30	0.99979	16.520	0.002
40	0.99998	21.314	0.000
50	1.00000	27.101	0.000
60	1.00000	32.387	0.000

(a)

SPRINGALL (1968) p.159

SPRINGALL MODEL

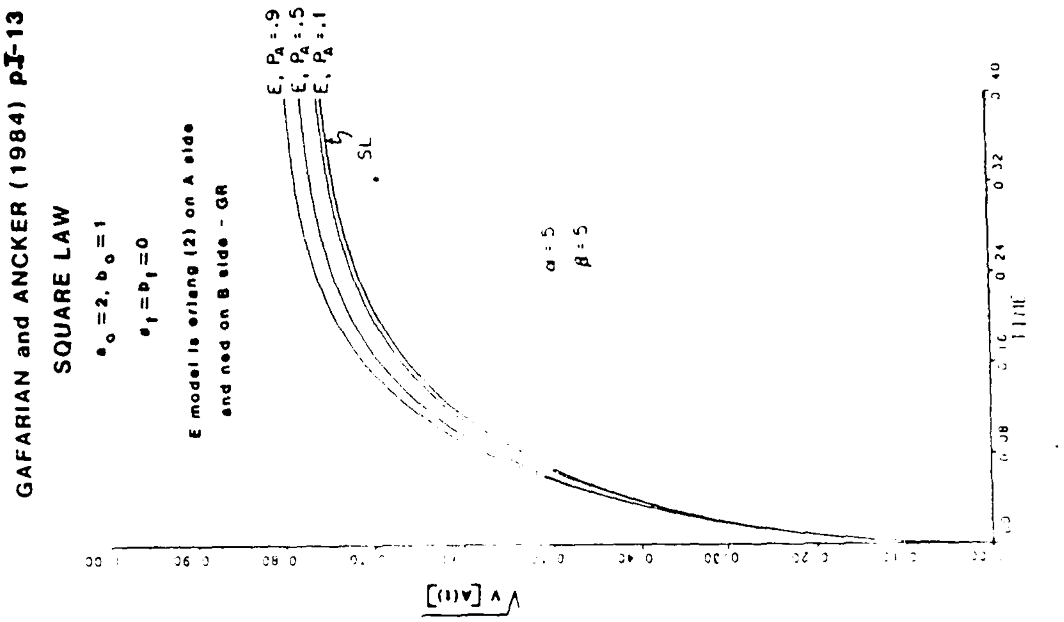
$a_0 = 50,\ b_0 = 40,\ m_2 = 50,\ \alpha = \beta = 1,\ \gamma = \delta = 10$

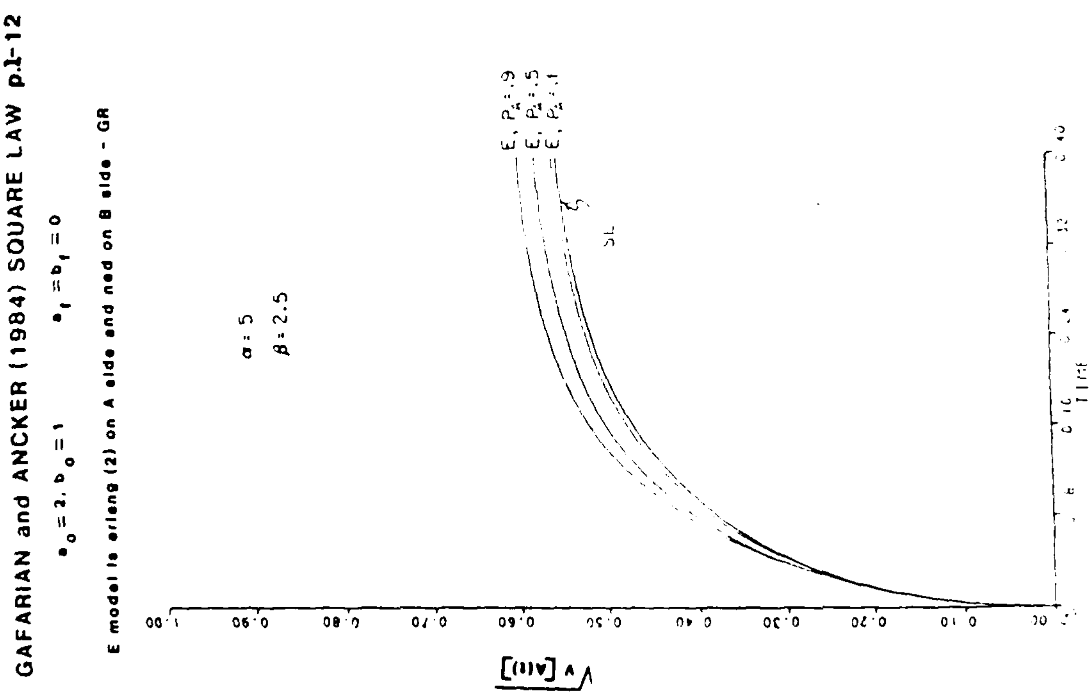
$a_f = b_f$	P(B)	V_B[B(∞)]
0	0.14455	4.770
1	0.14180	4.538
2	0.13900	4.311
4	0.13321	3.869
6	0.12715	3.446
8	0.12082	3.042
10	0.11420	2.659
20	0.07620	1.083
30	0.03071	0.192

(b)

Table V-6

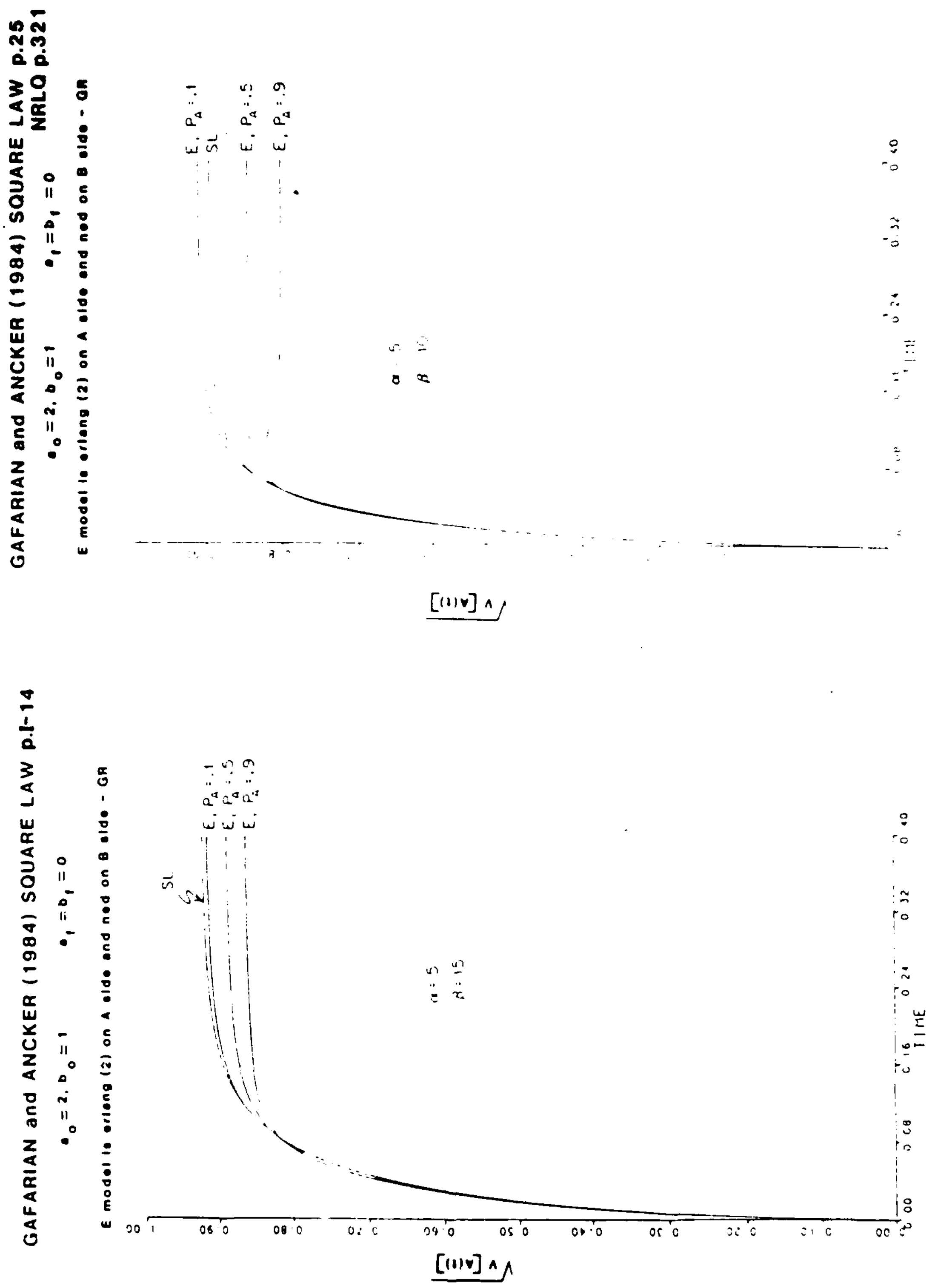
(b)


GAFARIAN and ANCKER (1984) p.I-13
SQUARE LAW
$a_o = 2, b_o = 1$
$a_1 = b_1 = 0$
E model is erlang (2) on A side
and ned on B side - GR
E, P_A = .9
E, P_A = .5
E, P_A = .1
SL
α = 5
β = 5
$\sqrt{V[A(t)]}$
0.80
0.40
0.30
0.20
TIME


GAFARIAN and ANCKER (1984) SQUARE LAW p.I-12
$a_o = 2, b_o = 1$
$a_1 = b_1 = 0$
E model is erlang (2) on A side and ned on B side - GR
α = 5
β = 2.5
E, P_A = .9
E, P_A = .5
E, P_A = .1
SL
$\sqrt{V[A(t)]}$
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
TIME

(a)

Figure V-2


GAFARIAN and ANCKER (1984) SQUARE LAW p.25
NRLQ p.321
a_o = 2, b_o = 1
a_f = b_f = 0
E model is erlang (2) on A side and ned on B side - GR
E, P_A : .1
SL
E, P_A : .5
E, P_A : .9
V [A(t)]
GAFARIAN and ANCKER (1984) SQUARE LAW p.I-14
a_o = 2, b_o = 1
a_f = b_f = 0
E model is erlang (2) on A side and ned on B side - GR
SL
E, P_A : .1
E, P_A : .5
E, P_A : .9
V [A(t)]
TIME
0.00
0.08
0.16
0.24
0.32
0.40
1.00
0.90
0.80
0.60
0.50
0.40
0.30

(b)

(a)

Figure V-3

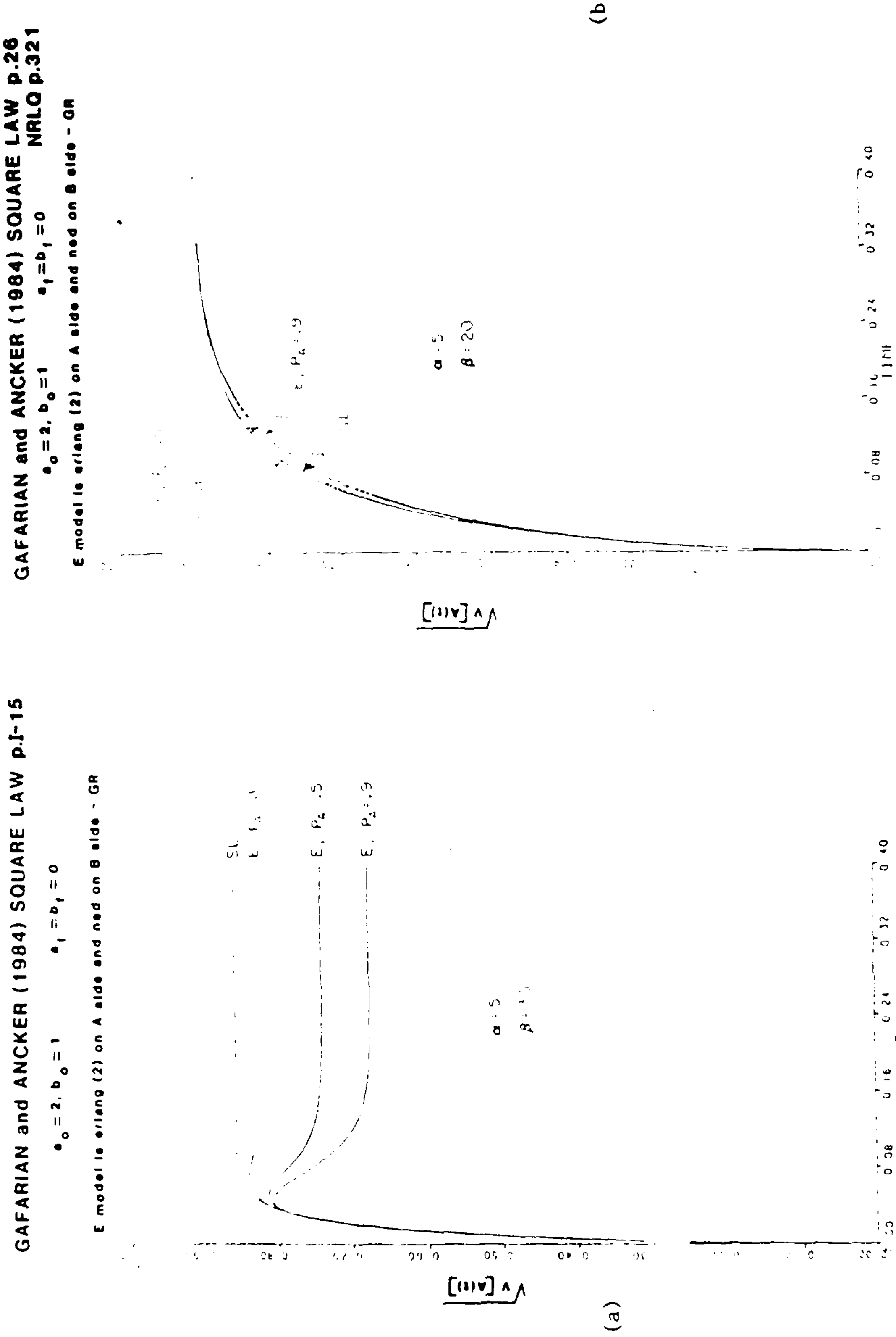

GAFARIAN and ANCKER (1984) SQUARE LAW p.26
NRLQ p.321
E model is erlang (2) on A side and ned on B side - GR
(b)
GAFARIAN and ANCKER (1984) SQUARE LAW p.I-15
E model is erlang (2) on A side and ned on B side - GR
(a)

Figure V-4

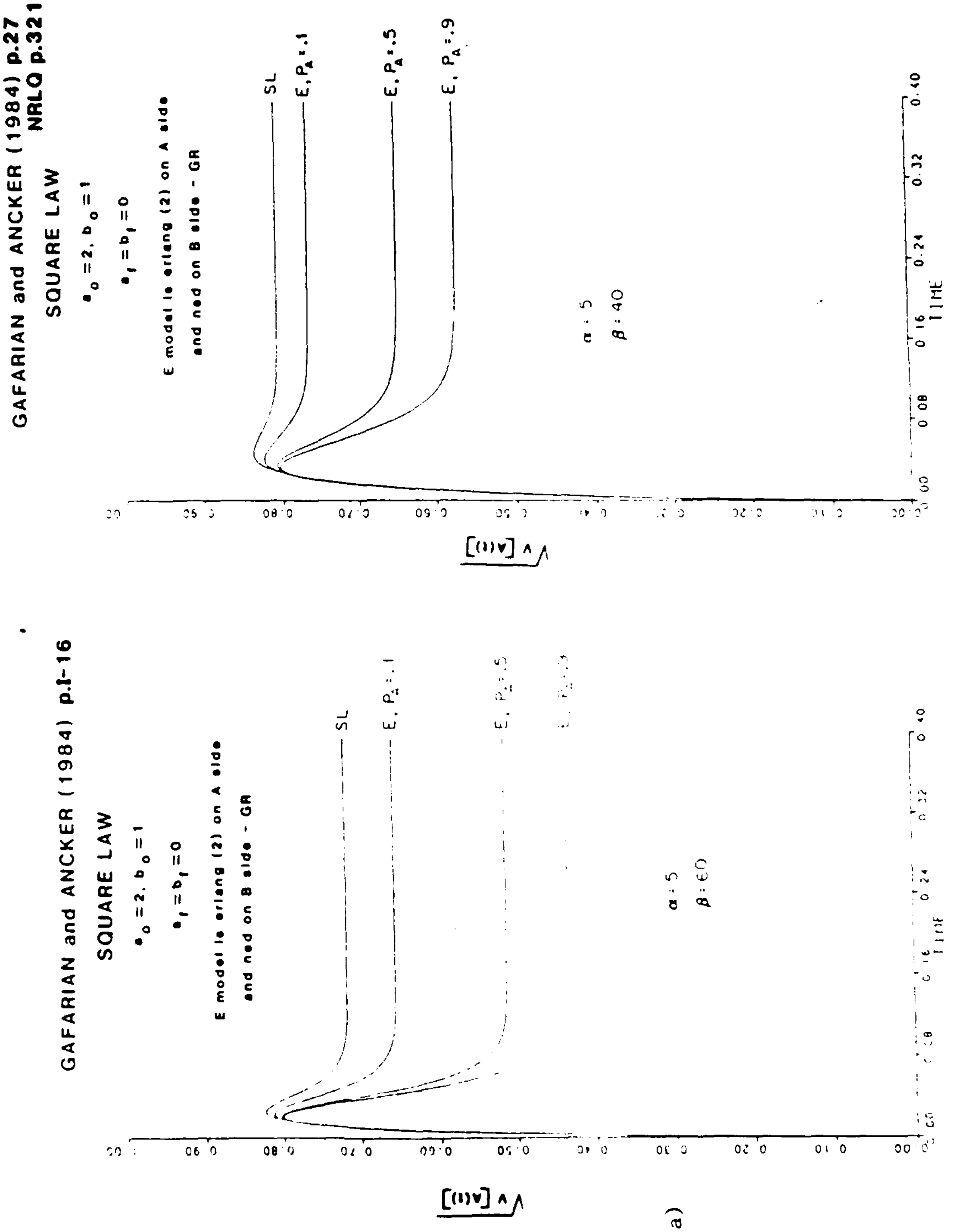
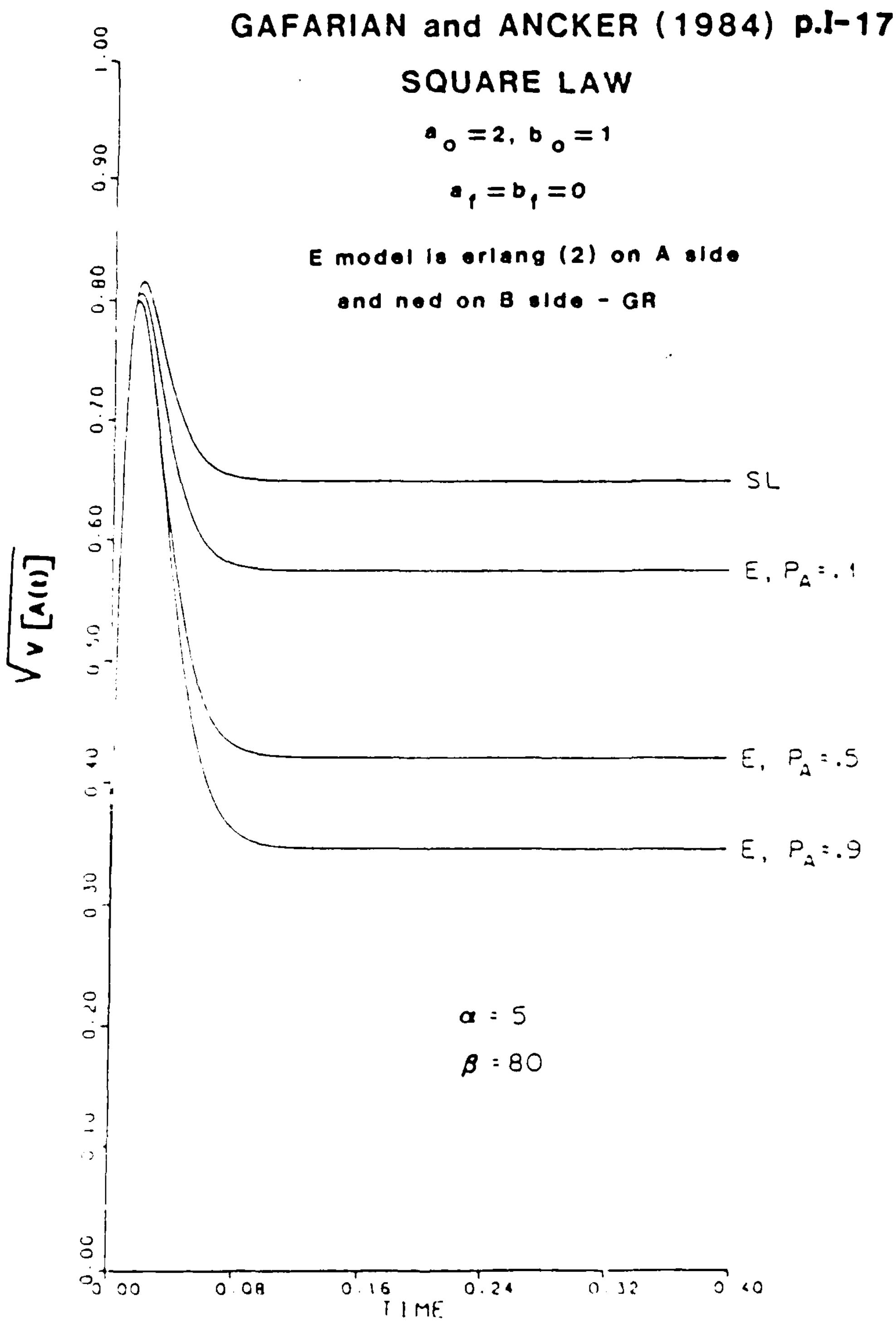

GAFARIAN and ANCKER (1984) p.I-16
SQUARE LAW
$a_o = 2,\ b_o = 1$
$a_1 = b_1 = 0$
E model is erlang (2) on A side
and ned on B side - GR
SL
$E, P_A = .1$
$E, P_A = .5$
$\alpha = 5$
$\beta = 60$
TIME
$\sqrt{V[A(t)]}$
(a)
GAFARIAN and ANCKER (1984) p.27
NRLQ p.321
SQUARE LAW
$a_o = 2,\ b_o = 1$
$a_1 = b_1 = 0$
E model is erlang (2) on A side
and ned on B side - GR
SL
$E, P_A = .1$
$E, P_A = .5$
$E, P_A = .9$
$\alpha = 5$
$\beta = 40$
TIME
$\sqrt{V[A(t)]}$
(b)

Figure V-5



GAFARIAN and ANCKER (1984) p.I-17
SQUARE LAW
$a_0 = 2,\ b_0 = 1$
$a_f = b_f = 0$
E model is erlang (2) on A side
and ned on B side - GR
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
α = 5
β = 80
$\sqrt{V[A(t)]}$
TIME
0.00
0.08
0.16
0.24
0.32
0.40
0.90
0.80
0.70
0.60
0.30
0.20

Figure V-6

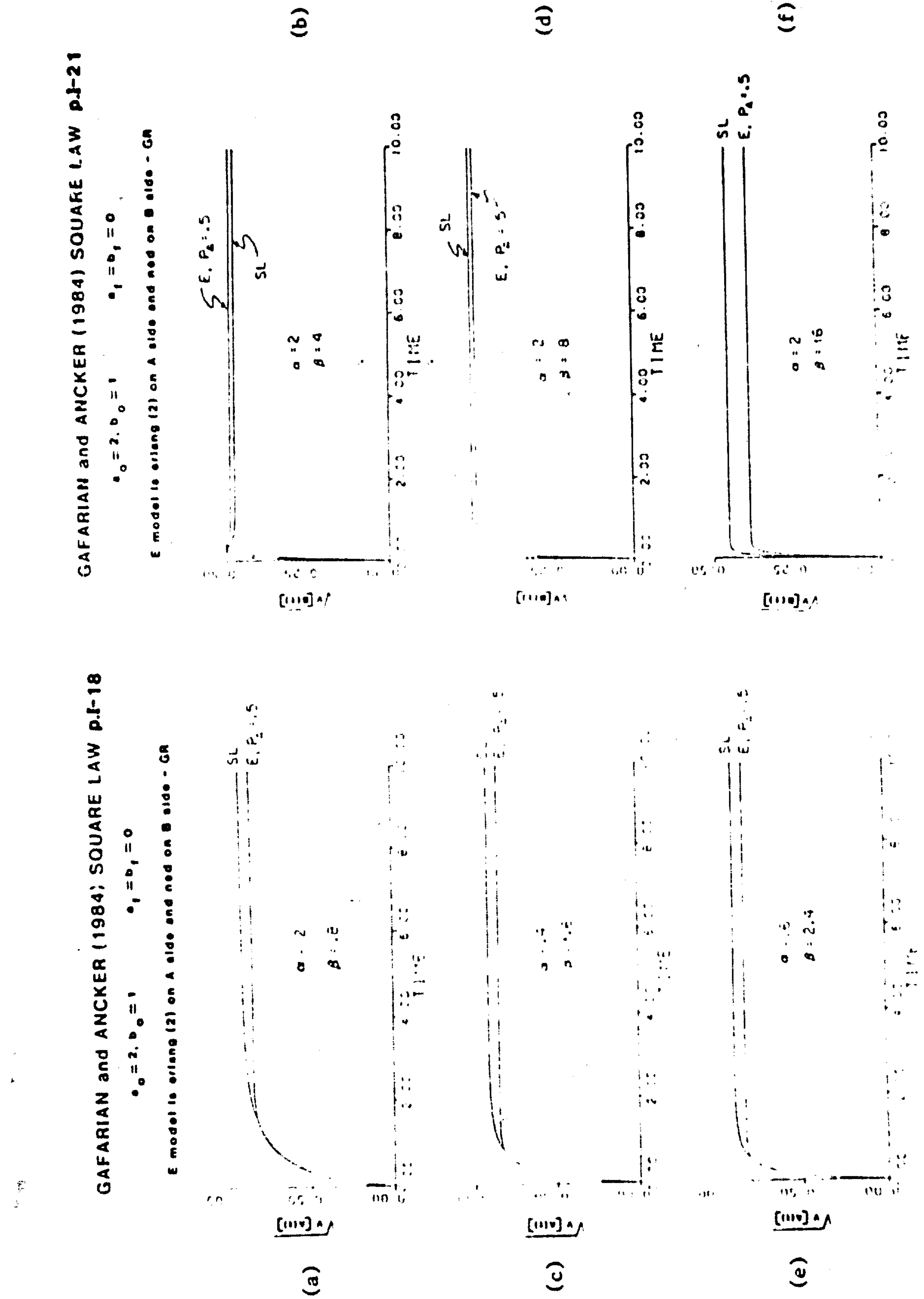

GAFARIAN and ANCKER (1984) SQUARE LAW p.I-18
a0 = 2, b0 = 1   a1 = b1 = 0
E model is erlang (2) on A side and ned on B side - GR
GAFARIAN and ANCKER (1984) SQUARE LAW p.I-21
a0 = 2, b0 = 1   a1 = b1 = 0
E model is erlang (2) on A side and ned on B side - GR
SL
E, P2 = .5
α = 2
β = 4
α = 2
β = 8
α = 2
β = 16
β = 2.4
TIME
2.00
4.00
6.00
8.00
10.00
0.25
0.50
(a)
(b)
(c)
(d)
(e)
(f)

Figure V-7

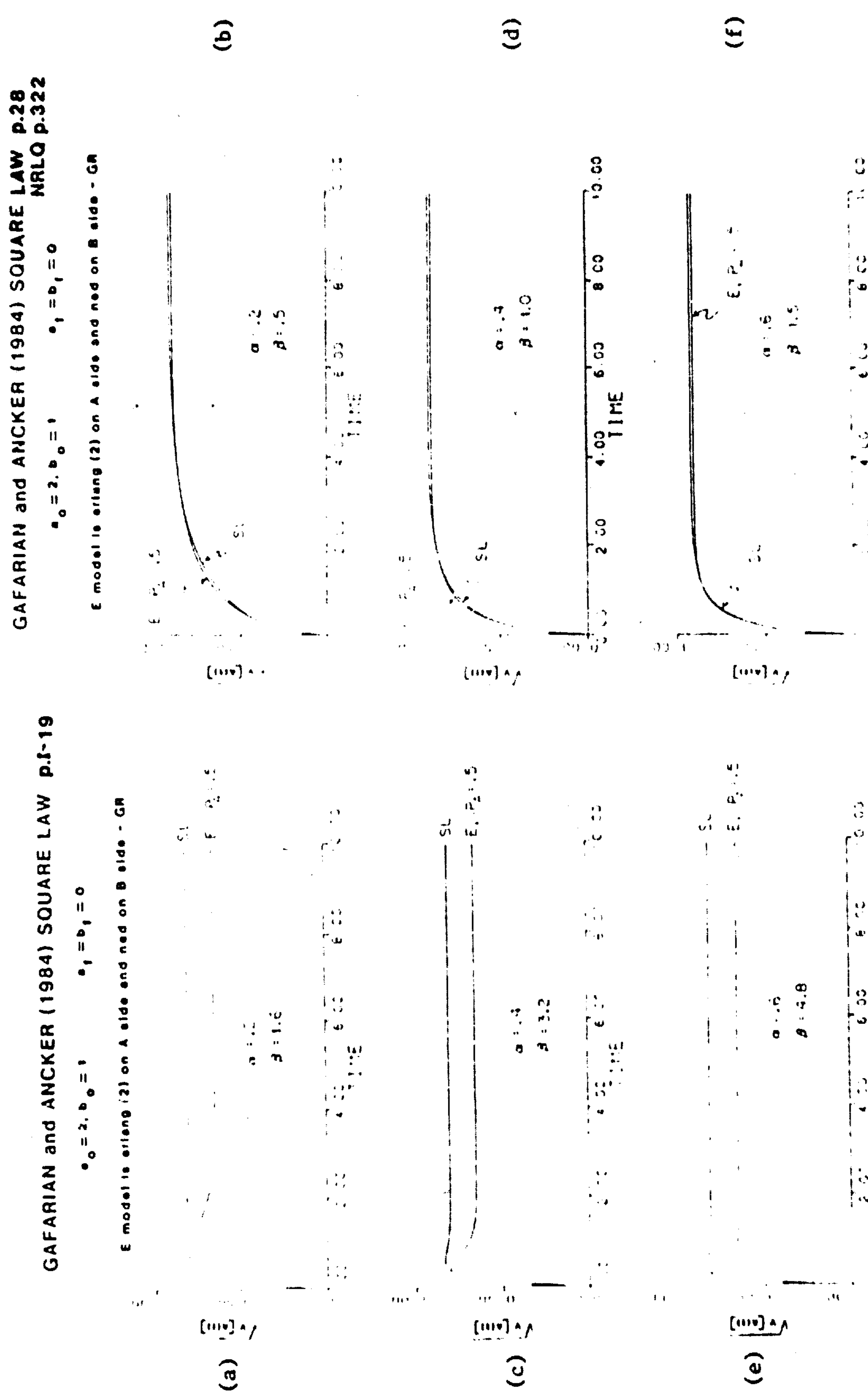

GAFARIAN and ANCKER (1984) SQUARE LAW p.I-19
a0 = 2, b0 = 1 a1 = b1 = 0
E model is erlang (2) on A side and ned on B side - GR
GAFARIAN and ANCKER (1984) SQUARE LAW p.28
NRLQ p.322
a0 = 2, b0 = 1 a1 = b1 = 0
E model is erlang (2) on A side and ned on B side - GR
(a)
(b)
(c)
(d)
(e)
(f)
SL
TIME

Figure V-8

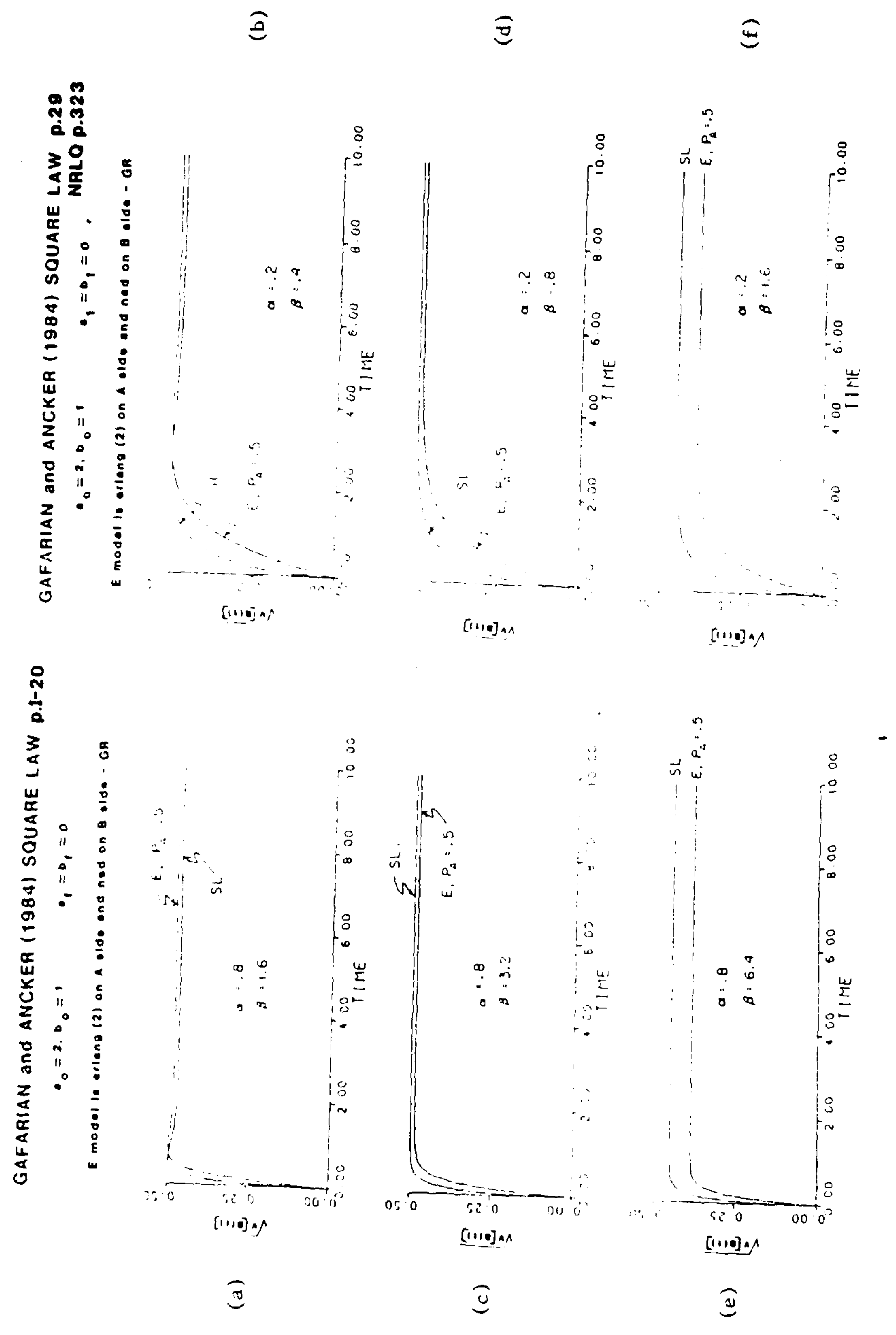
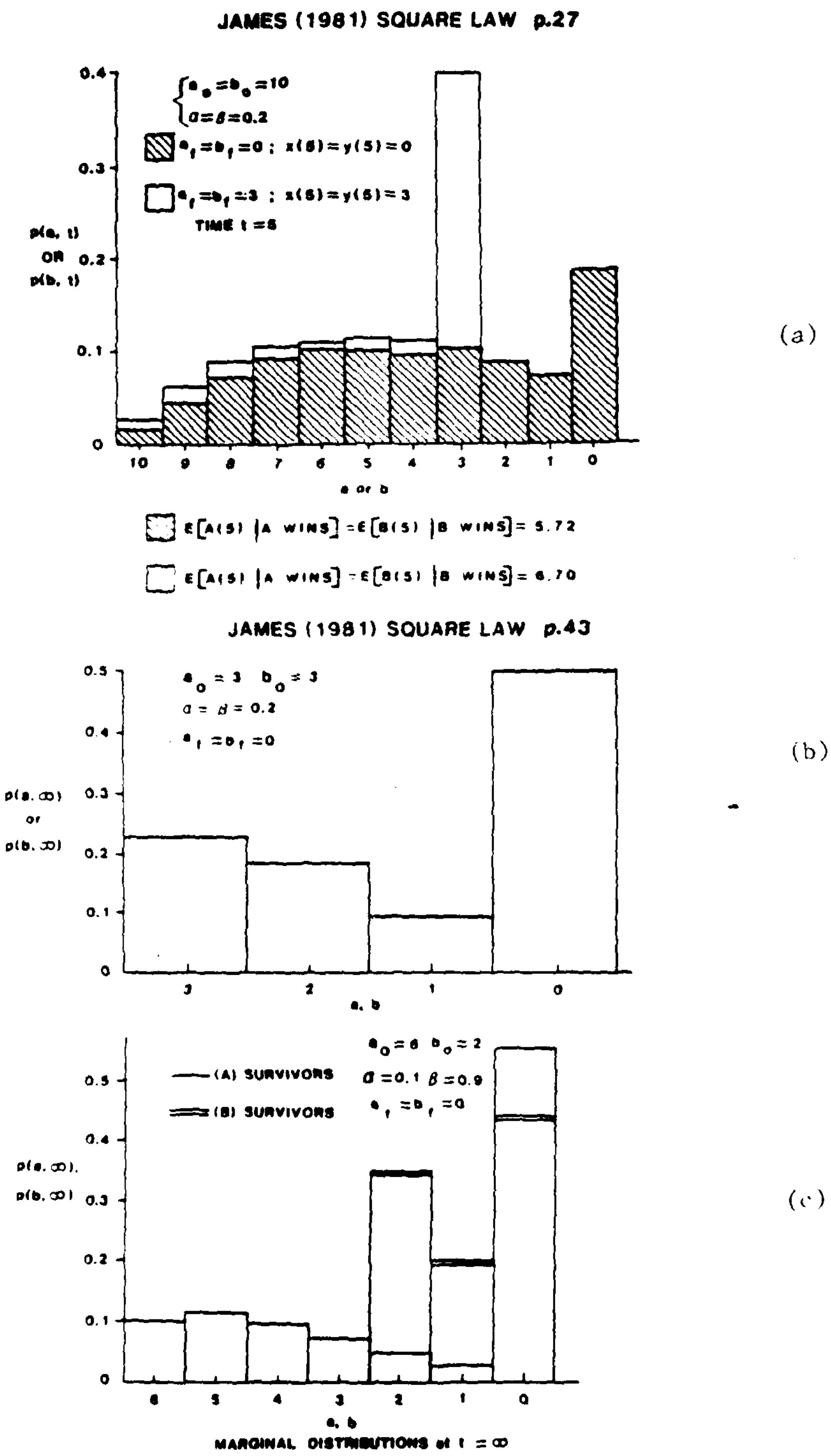

GAFARIAN and ANCKER (1984) SQUARE LAW p.I-20
GAFARIAN and ANCKER (1984) SQUARE LAW p.29
NRLQ p.323
E model is erlang (2) on A side and ned on B side - GR
TIME
SL
(a)
(b)
(c)
(d)
(e)
(f)

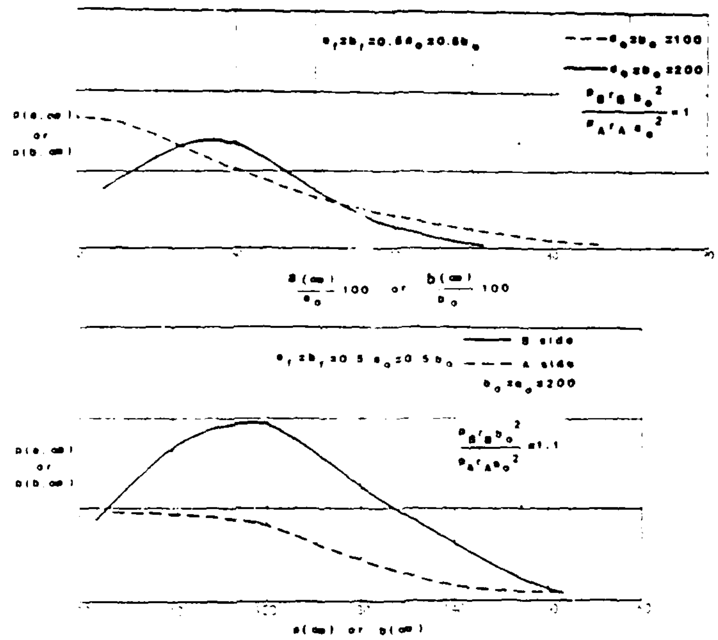
Figure V-9



JAMES (1981) SQUARE LAW p.27
(a)
a or b
JAMES (1981) SQUARE LAW p.43
(b)
a, b
(c)
(A) SURVIVORS
(B) SURVIVORS
a, b
MARGINAL DISTRIBUTIONS at t = ∞

Figure V-10

KARR (1975a) SQUARE LAW p.24

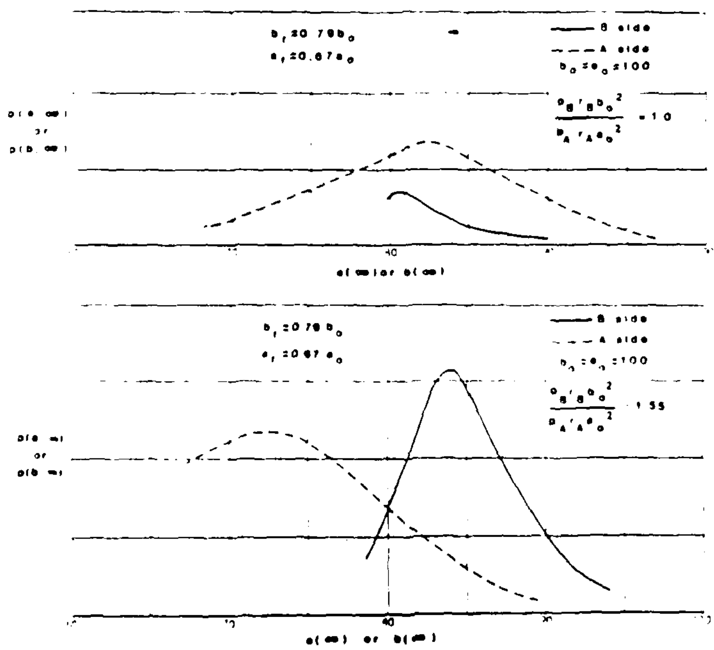


(a)

(b)

SCALE OF ORDINATES UNSPECIFIED

KARR (1975a) SQUARE LAW p.25

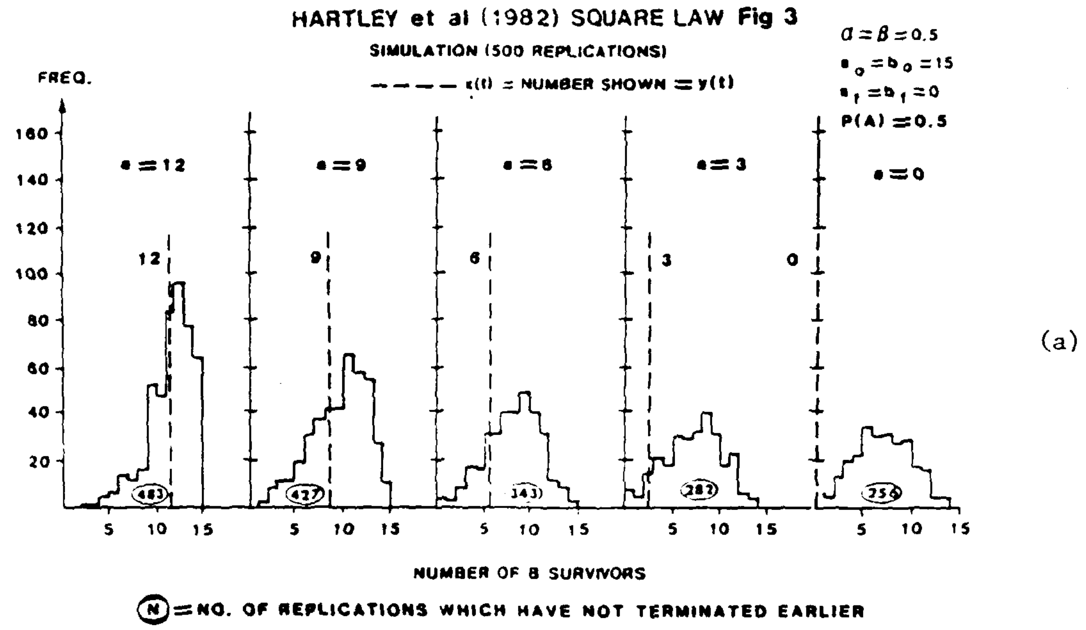


(c)

(d)

SCALE OF ORDINATES UNSPECIFIED

Figure V-11


HARTLEY et al (1982) SQUARE LAW Fig 3
SIMULATION (500 REPLICATIONS)
x(t) = NUMBER SHOWN = y(t)
FREQ.
NUMBER OF B SURVIVORS
N = NO. OF REPLICATIONS WHICH HAVE NOT TERMINATED EARLIER
(a)

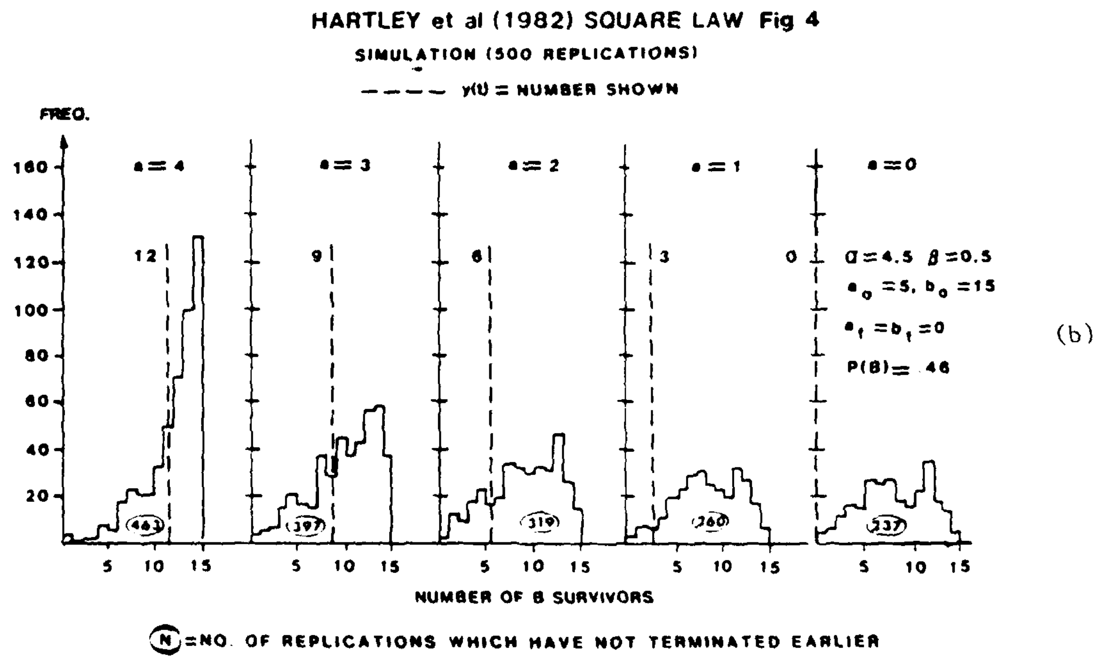
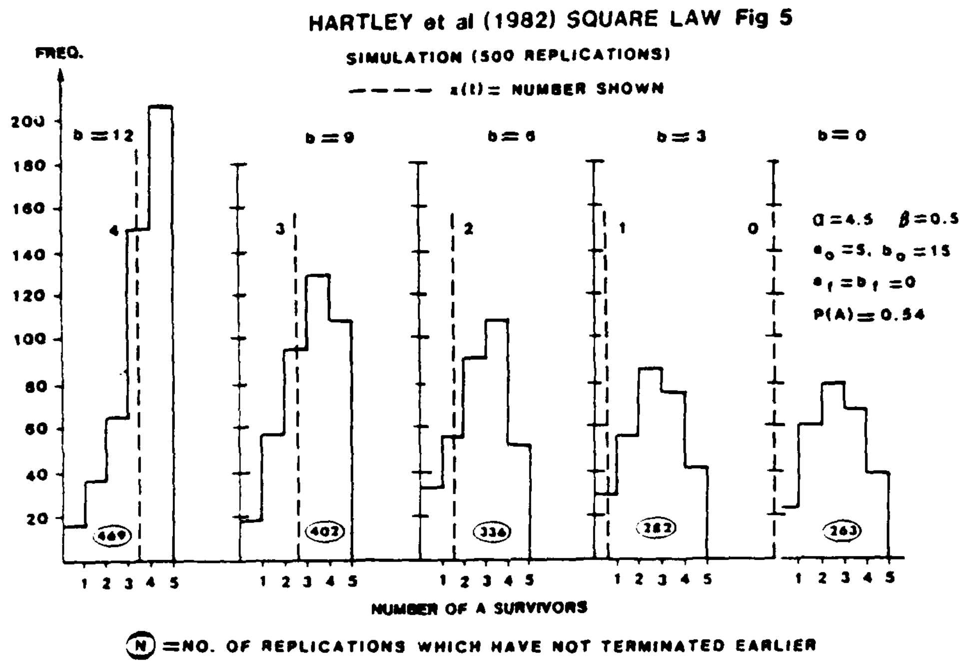
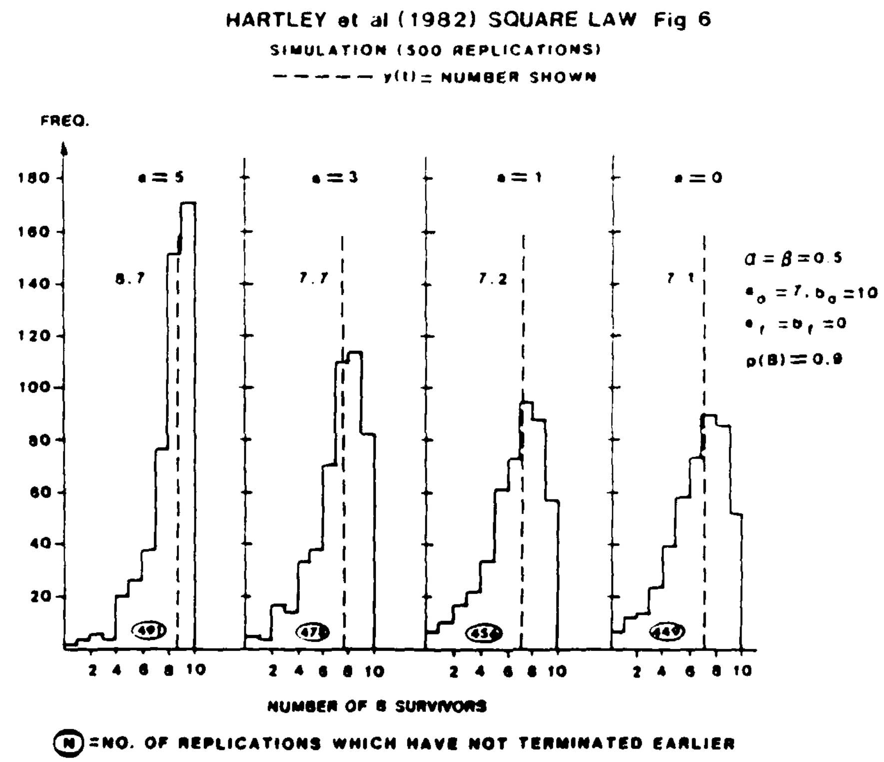

HARTLEY et al (1982) SQUARE LAW Fig 4
SIMULATION (500 REPLICATIONS)
y(t) = NUMBER SHOWN
FREQ.
NUMBER OF B SURVIVORS
N = NO. OF REPLICATIONS WHICH HAVE NOT TERMINATED EARLIER
(b)

Figure V-12.



(a)



(b)

Figure V-13.

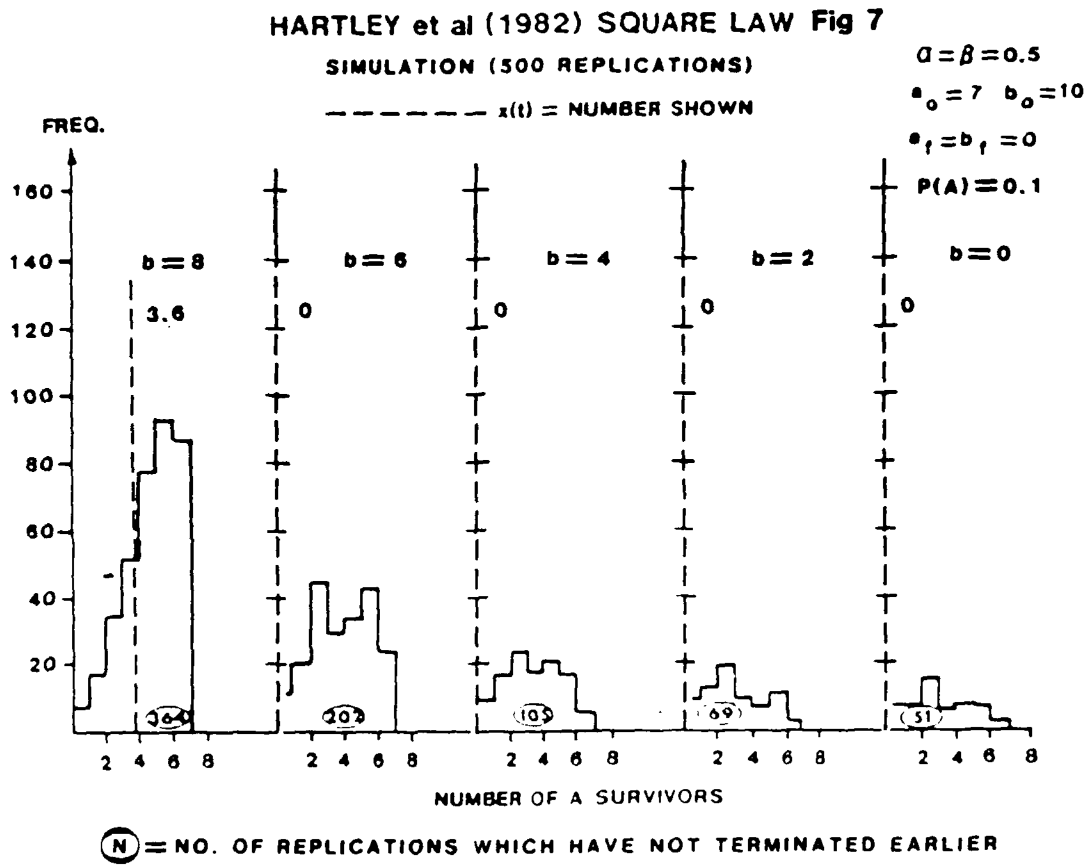

HARTLEY et al (1982) SQUARE LAW Fig 7
SIMULATION (500 REPLICATIONS)
------ x(t) = NUMBER SHOWN
$\alpha = \beta = 0.5$
$a_0 = 7$ $b_0 = 10$
$a_f = b_f = 0$
P(A) = 0.1
FREQ.
160
140
120
100
80
60
40
20
b = 8
3.6
b = 6
0
b = 4
0
b = 2
0
b = 0
0
2 4 6 8
NUMBER OF A SURVIVORS
N = NO. OF REPLICATIONS WHICH HAVE NOT TERMINATED EARLIER

Figure V-14.

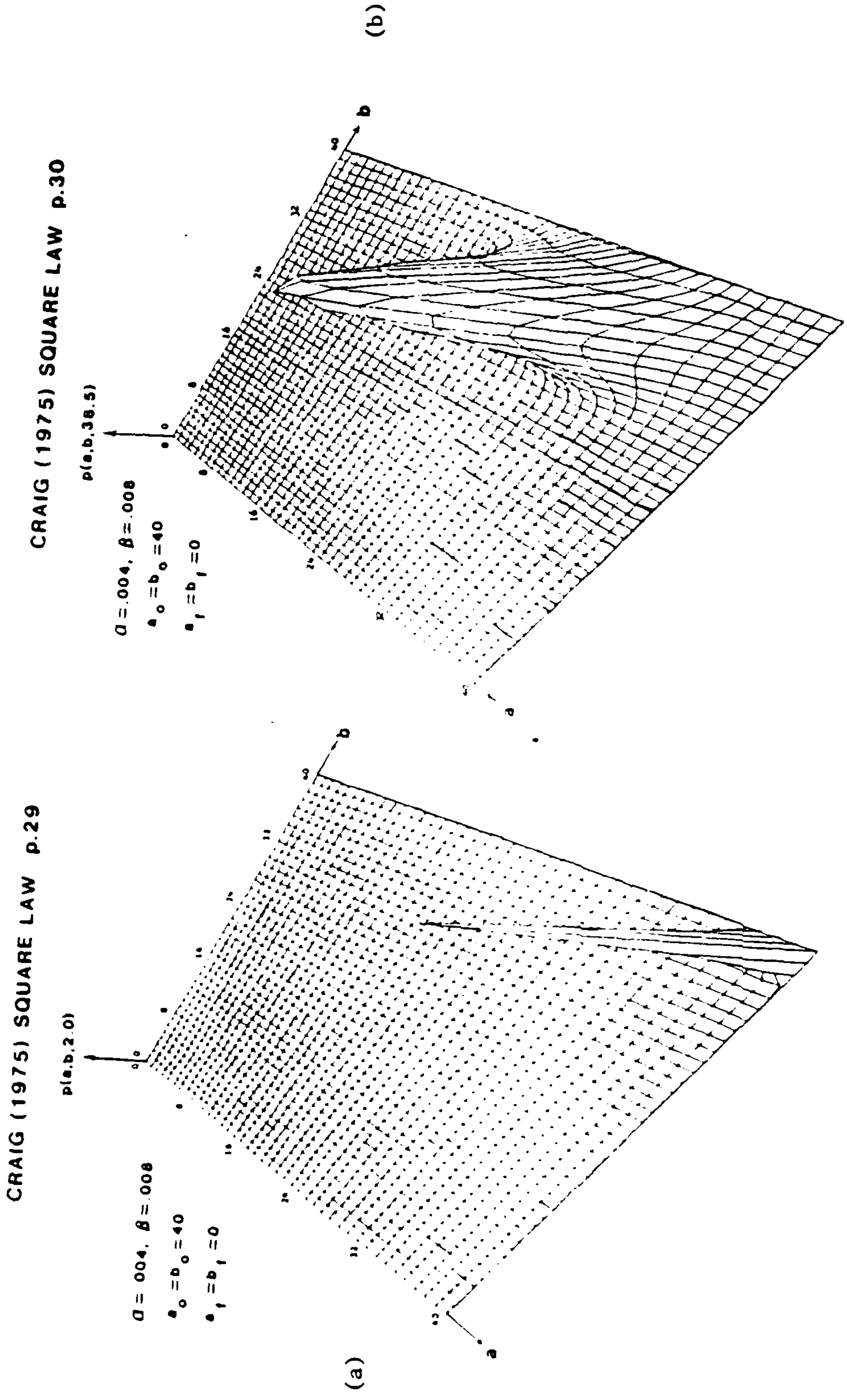

CRAIG (1975) SQUARE LAW p.29
p(a,b,2.0)
α = .004, β = .008
a₀ = b₀ = 40
a₁ = b₁ = 0
(a)
CRAIG (1975) SQUARE LAW p.30
p(a,b,38.5)
α = .004, β = .008
a₀ = b₀ = 40
a₁ = b₁ = 0
(b)

Figure V-15.

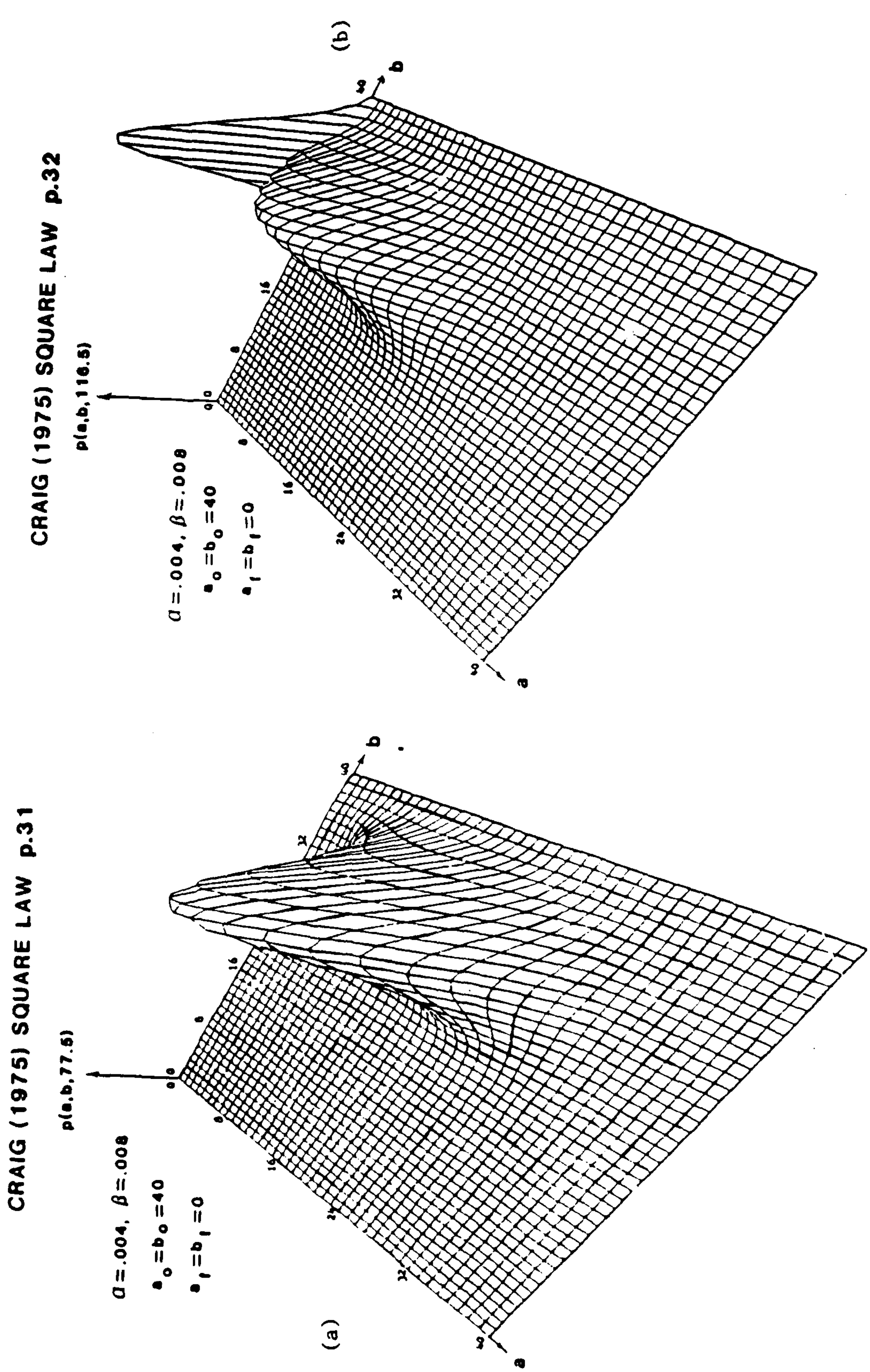

CRAIG (1975) SQUARE LAW p.32
p(a,b,116.5)
α=.004, β=.008
a_o=b_o=40
a_1=b_1=0
(b)
CRAIG (1975) SQUARE LAW p.31
p(a,b,77.5)
α=.004, β=.008
a_o=b_o=40
a_1=b_1=0
(a)

Figure V-16.

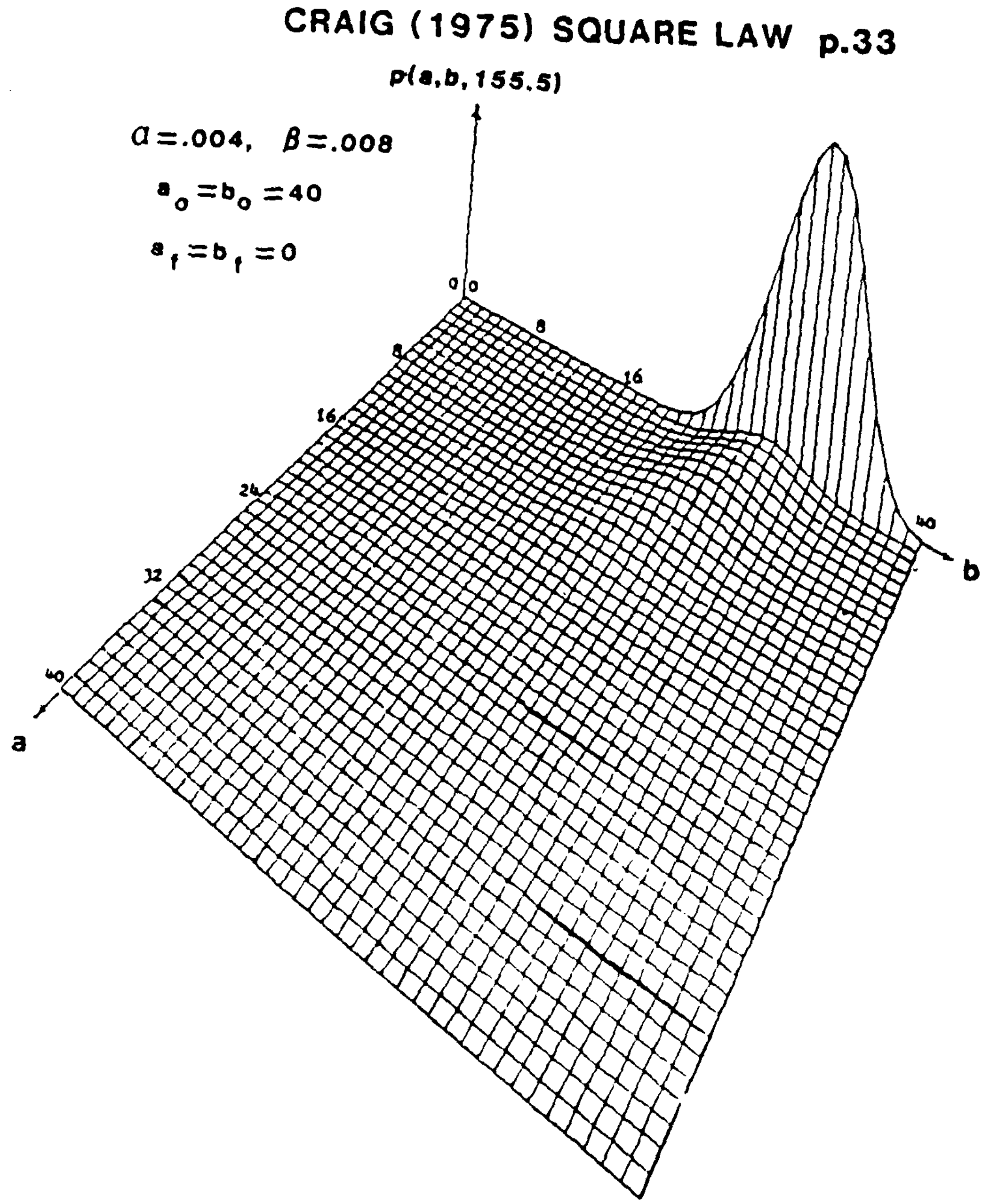

CRAIG (1975) SQUARE LAW p.33
p(a,b,155.5)
α=.004, β=.008
$a_0=b_0=40$
$a_f=b_f=0$
0
0
8
16
8
16
24
32
40
a
40
b

Figure V-17.

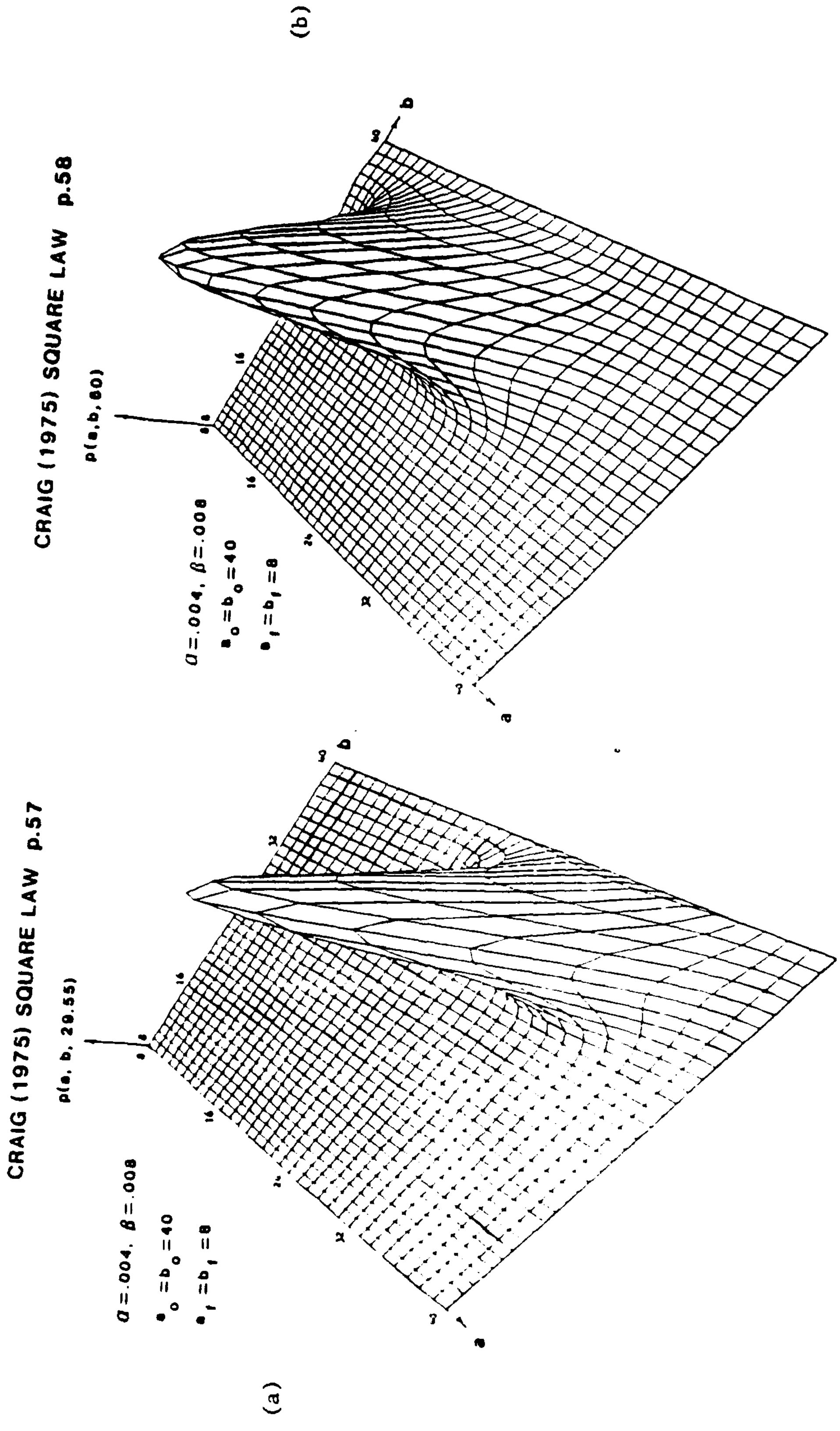

CRAIG (1975) SQUARE LAW p.57
p(a, b, 29.55)
α=.004, β=.008
a₀=b₀=40
a₁=b₁=8
(a)
CRAIG (1975) SQUARE LAW p.58
p(a,b,60)
α=.004, β=.008
a₀=b₀=40
a₁=b₁=8
(b)

Figure V-18.

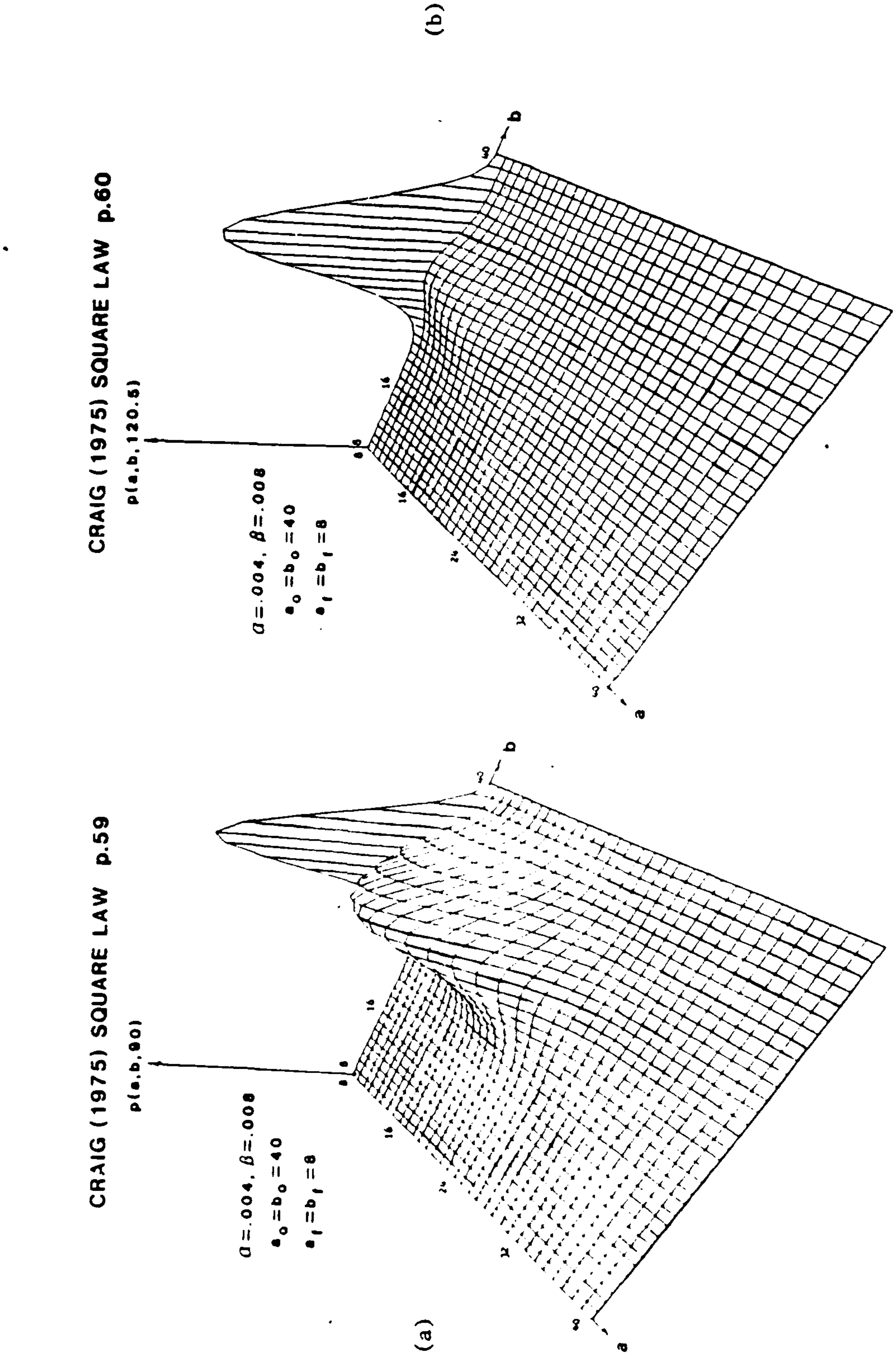
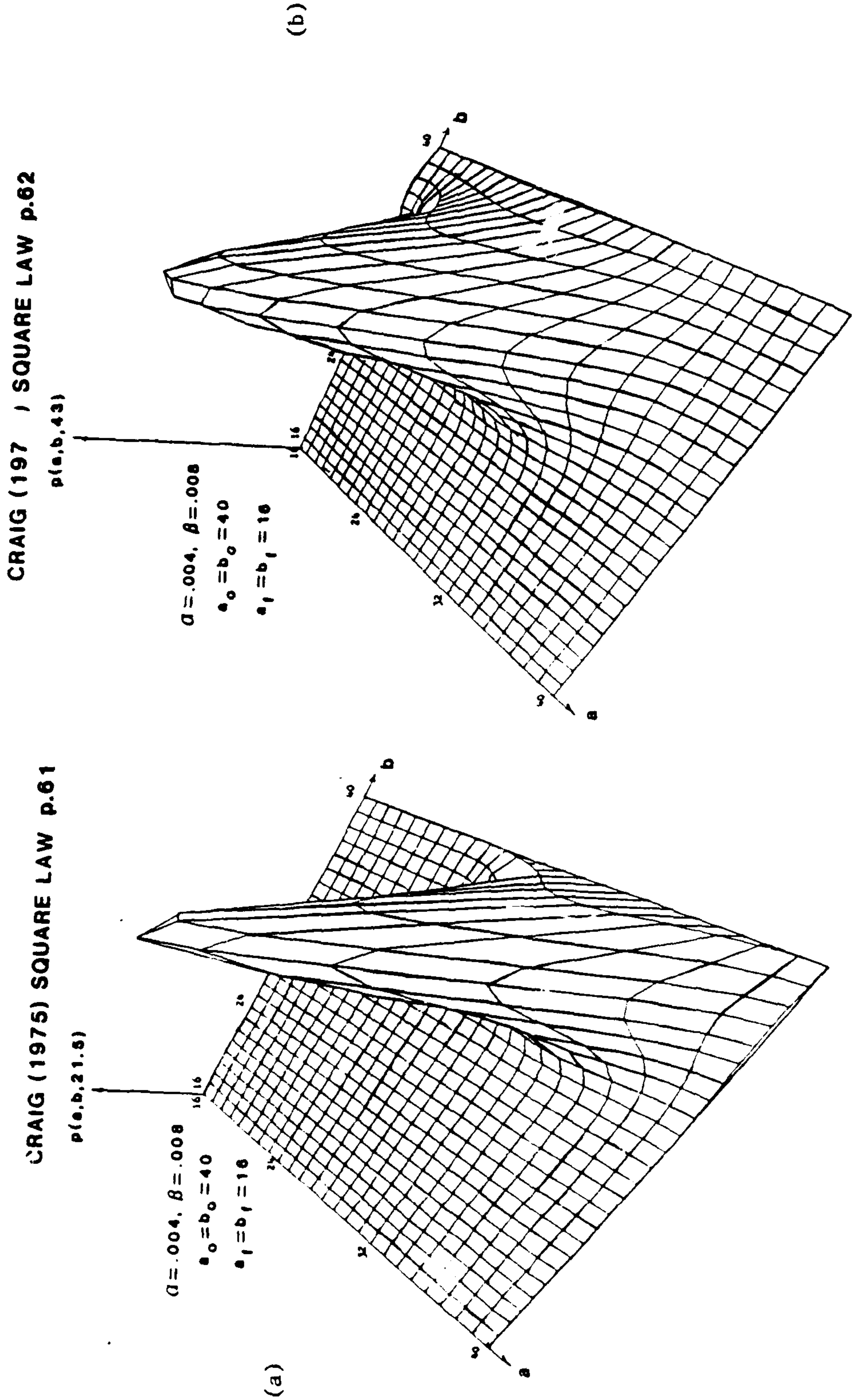

(a)
CRAIG (1975) SQUARE LAW p.59
p(a,b,90)
α=.004, β=.008
a₀=b₀=40
a₁=b₁=8
(b)
CRAIG (1975) SQUARE LAW p.60
p(a,b,120.5)
α=.004, β=.008
a₀=b₀=40
a₁=b₁=8

Figure V-19.



CRAIG (1975) SQUARE LAW p.61
p(a,b,21.5)
α=.004, β=.008
a₀=b₀=40
a₁=b₁=16
(a)
CRAIG (197 / SQUARE LAW p.62
p(a,b,43)
α=.004, β=.008
a₀=b₀=40
a₁=b₁=16
(b)

Figure V-20.

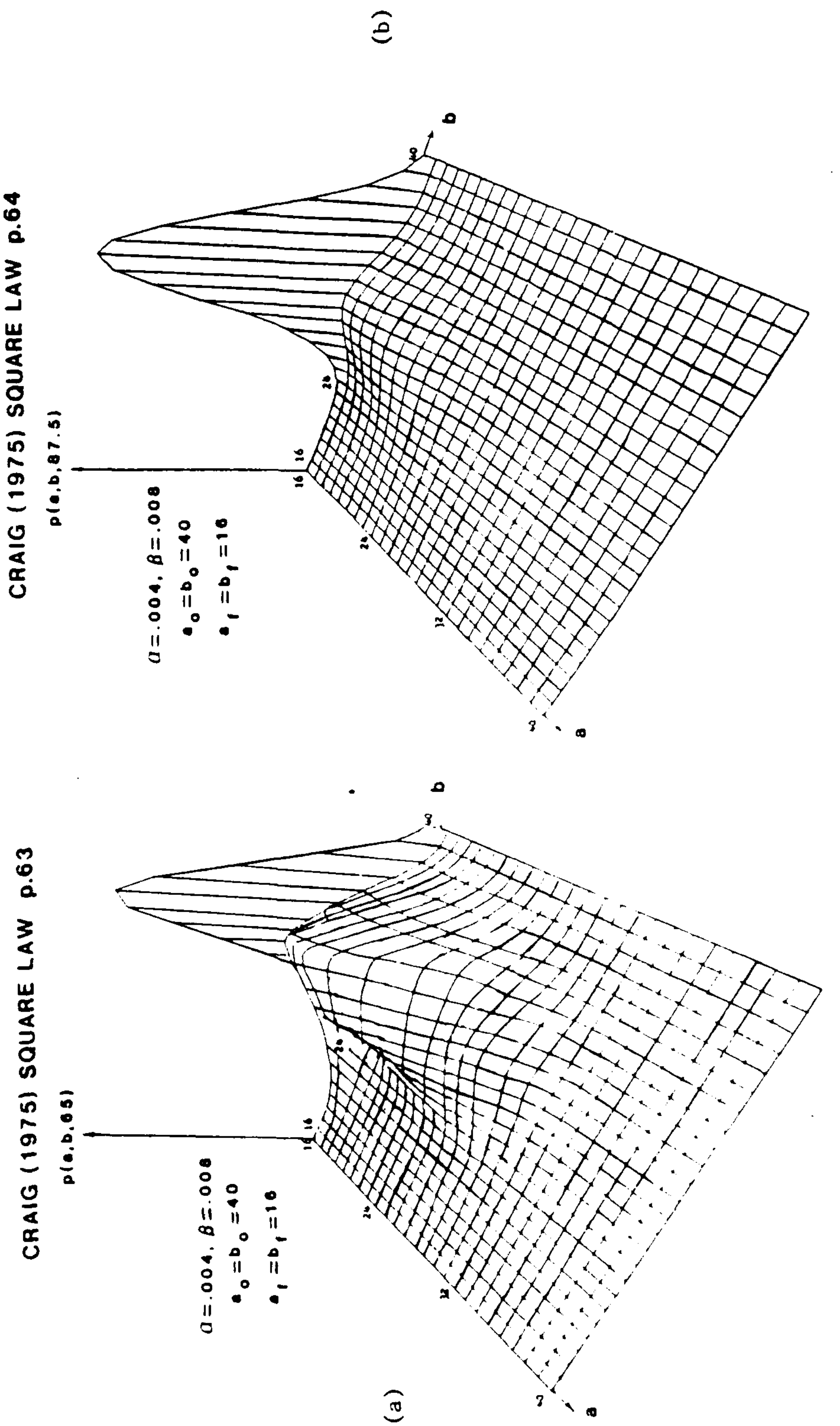

CRAIG (1975) SQUARE LAW p.63
p(a,b,65)
α=.004, β=.008
a₀=b₀=40
a_f=b_f=16
(a)
CRAIG (1975) SQUARE LAW p.64
p(a,b,87.5)
α=.004, β=.008
a₀=b₀=40
a_f=b_f=16
(b)

Figure V-21.

HARTLEY, HAGUES, AND KETTLE-WHITE (1982) p 10, p 12-A-11

SQUARE LAW

$a_0 = b_0 = 15$

$\alpha = \beta = 0.5$

$a_f = b_f = 0$

NUMBER OF B SURVIVORS	FREQUENCY FOR 500 REPLICATIONS	
	SIMULATION	THEORETICAL*
1	6	4
2	4	9
3	13	13
4	20	17
5	22	21
6	34	24
7	31	27
8	30	29
9	27	29
10	28	27
11	17	22
12	16	16
13	4	10
14	4	5
15	0	1

* Gye and Lewis (1976).

(a)

GYE AND LEWIS (1976) p 115

SQUARE LAW

$a_f = b_f = 0$

COMPARISON BETWEEN EXACT PROBABILITIES OF NUMBERS OF SURVIVORS AND APPROXIMATE VALUES AS CALCULATED FROM EQUATION (5)

•

BATTLE 1 : $a_0 = 10$, $b_0 = 4$, $\beta/\alpha = 25$ (B side superior)
BATTLE 2 : $a_0 = b_0 = 11$; $\beta/\alpha = 1$ (sides exactly matched)

WINNING SIDE WITH NO. OF SURVIVORS		WIN PROBABILITY EXACT	WIN PROBABILITY APPROX
BATTLE 1: FOR L, P(B) = 1.0			
B with	1	0.588	0.578
	2	0.275	0.263
	3	0.089	0.089
	4	0.025	0.030
TOTAL		0.977	0.961
A with	1	0.001	0.001
	2	0.002	0.002
	3	0.003	0.003
	4	0.003	0.004
	5	0.004	0.005
	6	0.004	0.005
	7	0.003	0.005
	8	0.002	0.004
	9	0.001	0.004
	10	0.001	0.006
TOTAL		0.023	0.039

WINNING SIDE WITH NO. OF SURVIVORS		PROBABILITY EXACT	PROBABILITY APPROX
BATTLE 2: FOR L, P(B) = 0 = P(A)			
B with	1	0.013	0.013
	2	0.027	0.026
	3	0.040	0.039
	4	0.053	0.052
	5	0.064	0.064
	6	0.071	0.072
	7	0.073	0.075
	8	0.067	0.069
	9	0.052	0.052
	10	0.030	0.029
	11	0.010	0.009
TOTAL		0.500	0.500

(b)

Table V-7.

GYE AND LEWIS (1974) p 15

SQUARE LAW

$a_0 = 50,\ b_0 = 100$

$a_f = b_f = 0$

$\alpha = \beta$

(a)

NUMBER OF SURVIVORS (b)	p(o, b, ∞)		NUMBER OF SURVIVORS (b)	p(o, b, ∞)	
	EXACT	APPROXIMATE From Eq (31)		EXACT	APPROXIMATE From Eq (31)
100		0.0002	80	0.0302	0.0307
99	0.0002	0.0005	79	0.0232	0.0234
98	0.0007	0.0014	78	0.0174	0.0174
97	0.0023	0.0033	77	0.0128	0.0128
96	0.0058	0.0069	76	0.0093	0.0092
95	0.0120	0.0128	75	0.0066	0.0065
94	0.0213	0.0215	74	0.0046	0.0045
93	0.0334	0.0326	73	0.0032	0.0031
92	0.0473	0.0456	72	0.0022	0.0021
91	0.0613	0.0591	71	0.0015	0.0014
90	0.0735	0.0712	70	0.0010	0.0009
89	0.0824	0.0806	69	0.0006	0.0006
88	0.0870	0.0860	68	0.0004	0.0004
87	0.0873	0.0871	67	0.0003	0.0003
86	0.0835	0.0840	66	0.0002	0.0002
85	0.0766	0.0776	65	0.0001	0.0001
84	0.0677	0.0688	64	0.0001	0.0001
83	0.0578	0.0589	63		0.0001
82	0.0478	0.0488			
81	0.0385	0.0392			

GYE AND LEWIS (1974) p 16

SQUARE LAW

$a_0 = b_0 = 40$

$a_f = b_f = 0$

$\alpha = \beta$

(b)

NUMBER OF SURVIVORS (b)	p(o, b, ∞)		NUMBER OF SURVIVORS (b)	p(o, b, ∞)	
	EXACT	APPROXIMATE From Eq (31)		EXACT	APPROXIMATE From Eq (31)
40			20	0.0252	0.0251
39			19	0.0260	0.0259
38			18	0.0265	0.0264
37		0.0001	17	0.0265	0.0264
36	0.0001	0.0002	16	0.0262	0.0261
35	0.0002	0.0003	15	0.0256	0.0255
34	0.0005	0.0007	14	0.0246	0.0245
33	0.0010	0.0012	13	0.0235	0.0234
32	0.0017	0.0019	12	0.0221	0.0220
31	0.0028	0.0030	11	0.0205	0.0205
30	0.0042	0.0044	10	0.0189	0.0188
29	0.0060	0.0062	9	0.0171	0.0171
28	0.0081	0.0082	8	0.0153	0.0153
27	0.0105	0.0106	7	0.0134	0.0134
26	0.0130	0.0130	6	0.0115	0.0115
25	0.0155	0.0155	5	0.0096	0.0096
24	0.0180	0.0179	4	0.0077	0.0077
23	0.0202	0.0202	3	0.0058	0.0058
22	0.0222	0.0222	2	0.0039	0.0038
21	0.0239	0.0238	1	0.0019	0.0019

NOTE: Values in the table are also $p(a, o, \infty)$ for a survivors.

Table V-8.

WEALE (1976) pp.69, 70

SQUARE LAW

$$a_0 = b_0 = 10$$

$$a_f = 2, \; b_f = 3$$

$$\alpha = 0.5, \; \beta = 0.25$$

$$P(A) = 0.776303, \; P(B) = 0.030909, \; P(D) = 0.192788$$

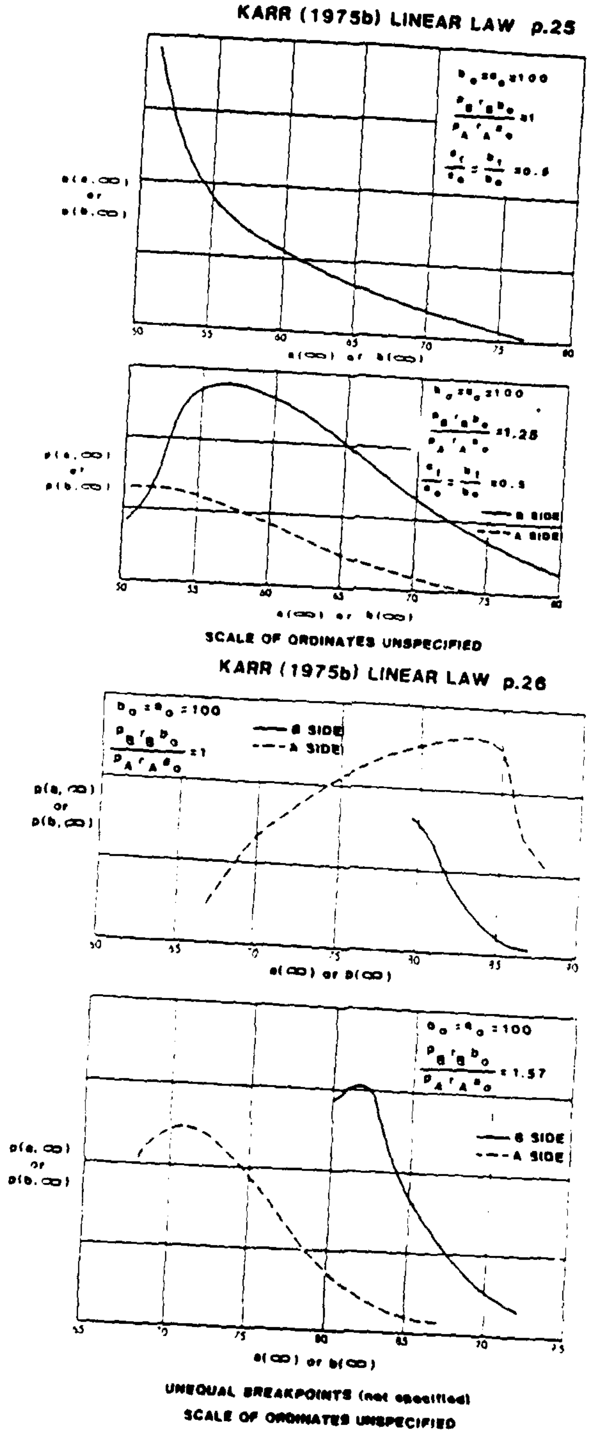
DRAWS OCCUR IF:

$a = 2, \; b \leq 6$ or
$b = 3, \; a \leq 5$

$$p(a,b,\infty) \times 1{,}000{,}000$$

A VICTORY			B VICTORY		
b	a	P	a	b	P
3	10	124,751	2	10	1,506
3	9	199,438	2	9	5,092
3	8	194,456	2	8	9,882
3	7	151,938	2	7	14,429
3	6	105,720			
DRAW LEVEL			**DRAW LEVEL**		
b	a	P	a	b	P
3	5	69,044	2	6	17,551
3	4	43,447	2	5	18,646
3	3	26,459	2	4	17,640

Table V-9



(a)

(b)

(c)

(d)

Figure V-22.

WEALE (1975) pp. 64,65

SPECIAL MODEL

$a_0 = b_0 = 10$

$a_f = 2,\ b_f = 3$

$P(A) = .992809$

$P(B) = .000018$

$P(D) = .007173$

DRAWS OCCUR IF:

$a = 2,\ b \leq 6$
or
$b = 3,\ a \leq 5$

ATTRITION COEFFICIENTS:

A SIDE = $a(0.05 + 0.05b)$

B SIDE = $b(0.025 + .005a)$

$p(a,b,\infty) \times 1{,}000{,}000$

A VICTORY			B VICTORY		
b	a	P	a	b	P
3	10	425,276	2	10	0
3	9	336,501	2	9	1
3	8	156,656	2	8	5
3	7	56,612	2	7	12
3	6	17,764			

DRAW LEVEL			DRAW LEVEL		
b	a	P	a	b	P
3	5	5,156	2	6	27
3	4	1,446	2	5	52
3	3	406	2	4	87

Table V-10

SPRINGALL (1968) p.170

SPRINGALL MODEL

$a_0 = 10,\ b_0 = 10,\ m_2 = 5,\ \alpha = 0.9,\ \beta = 1,\ \gamma = 2,\ \delta = 2$

$a_f = b_f = 0$

b	$p(o,b,\infty)$
1	0.09486
2	0.09809
3	0.09547
4	0.08639
5	0.07147
6	0.05279
7	0.03360
8	0.01737
9	0.00653
10	0.00135

(a)

SPRINGALL (1968) p.182

SPRINGALL MODEL

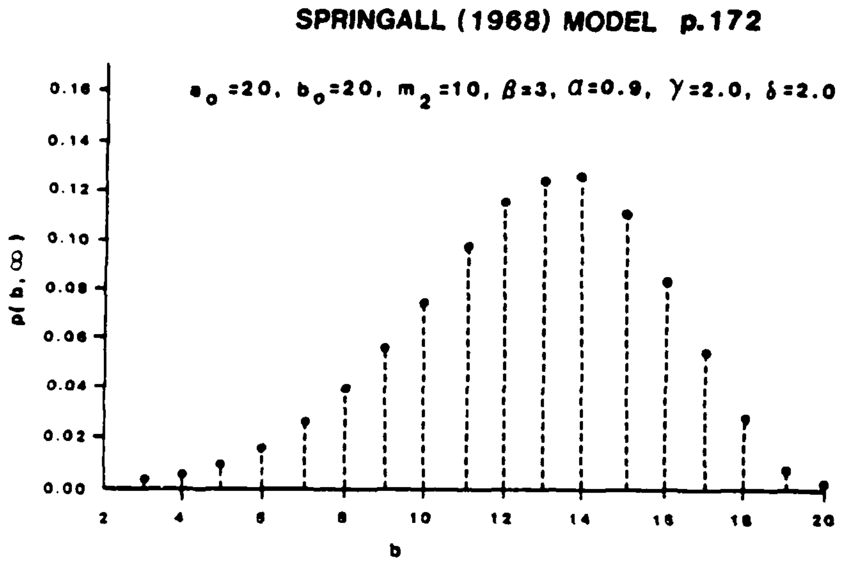
$a_0 = 15,\ b_0 = 20,\ m_2 = 5,\ n_2 = 5,\ \alpha = 0.90,\ \beta = 1.00,\ \gamma = 2.00,\ \delta = 2.00$

$a_f = b_f = 0$

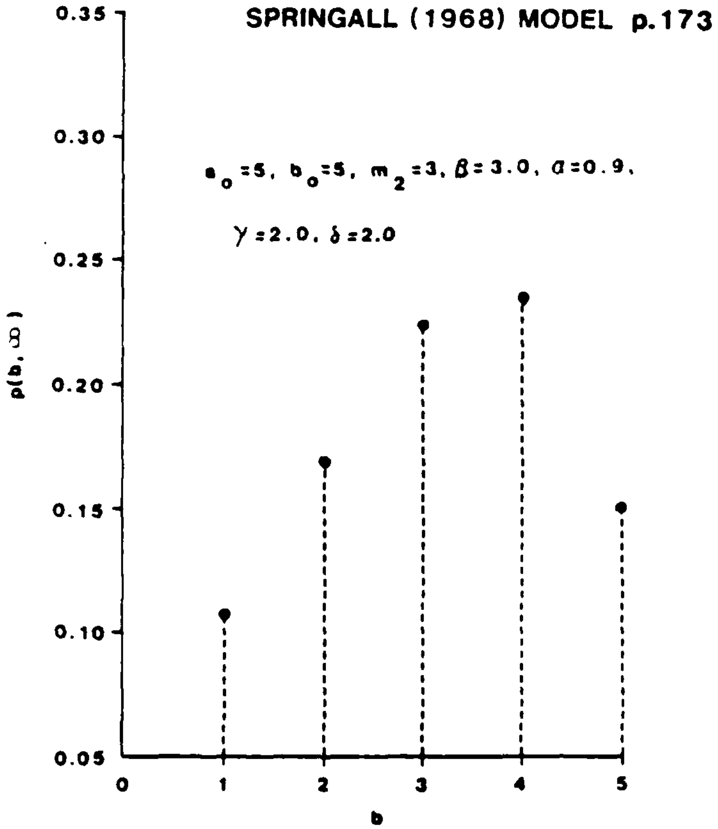
P(B) = 0.85503		P(A) = 0.14497	
b	$p(o,b,\infty)$	a	$p(a,o,\infty)$
1	0.04072	1	0.03936
2	0.04855	2	[illegible]
3	0.05657	3	0.02431
4	0.06424	4	0.01736
5	0.07096	5	[illegible]
6	0.07602	6	0.00839
7	0.07872	7	[illegible]
8	0.07850	8	0.00297
9	0.07504	9	0.00151
10	0.06838	10	0.00071
11	0.05901	11	0.00029
12	0.04783	12	0.00010
13	0.03602	13	0.00003
14	0.02487	14	0.00000
15	0.01545	15	0.00000
16	0.00842		
17	0.00388		
18	0.00142		
19	0.00037		
20	0.00005		

(b)

Table V-11



(a)



(b)

Figure V-23.

VI. MISCELLANEOUS RESULTS

Two kinds of results are presented in this section. They are: (1) examples of correlation between A and B survivors, and (2) some new results on combining small fire-fights to describe large battles.

In Figures VI-1 and VI-2 and Tables VI-1 through VI-7, the figures provide visual evidence of correlation (or equivalently covariance) and the tables give actual computed values. The significance of this is that covariance (correlation) is the term whose existence (and magnitude) causes the Linear Law and any model in which it is a component (e.g. the Weale Special Model and the Springall Model) to have differing L and SL mean value functions. Other consequences of correlation have not been explored.

In the remaining part, (figures VI-3 through VI-12), two situations are examined. Both include comparisions of L with SL fire-fights in which a large number of combatants either engage in a standard Square Law battle or the same total initial numbers engage in several identical, simultaneous and independent Square Law fire-fights. The two situations are: (1) 48 on each side for the large battle and 8 smaller battles with 6 on each side and; (2) 96 on each side for the large battle and 16 smaller fire-fights with 6 on each side. The principal results are that the L model has identical outcomes no matter how the battle is sub-divided, but the large SL battles differ considerably from the several smaller battles. This is true both for the mean value functions and the standard deviations. This substantiates the position that the compartmentalized nature of real combat cannot be safely ignored. This is some very early work on the problem. Much more needs to be done before any clear picture of the seriousness of this situation will emerge.

PERLA AND LEHOCZKY (1977) p.27

SQUARE LAW

$a_f = b_f = 0$

$\alpha = \beta = .05$

$t = 15$

CORRELATION [A(t) B(t)]			
(a_0,b_0)	S	D	|S-D|/S
(20,20)	-.594 (.005)	-.686	.0115
(25,25)	-.684 (.015)	-.686	.0029
(30,30)	-.683 (.001)	-.686	.0044
(40,40)	-.689 (.006)	-.686	.0044
(50,50)	-.687 (.009)	-.686	.0015

S = Simulation, 6000 replications. Numbers in parentheses are standard deviations of the S estimates.

D = Perla & Lehoczky (1977) diffusion approximation.

(a)

PERLA AND LEHOCZKY (1977) p.28

SQUARE LAW

$a_f = b_f = 0$

$\alpha = .075,\ \beta = .030$

$t = 10$

CORRELATION [A(t) B(t)]			
(a_0,b_0)	S	D	|S-D|/S
(50,20)	-.512 (.011)	-.526	.0273
(75,30)	-.534 (.008)	-.526	.0150
(100,40)	-.547 (.023)	-.526	.0384
(125,50)	-.518 (.038)	-.526	.0154
(250,100)	-.527 (.006)	-.526	.0019

S = Simulation, 6000 replications. Numbers in parentheses are standard deviations of the S estimates.

D = Perla & Lehoczky (1977) diffusion approximation.

(b)

Table VI-1.

WEALE (1972) pp.49,50

SQUARE LAW

$a_0 = b_0 = 10$

$a_f = b_f = 0$

$\alpha = 0.05, \quad \beta = 0.025$

T	Cov[A(t) B(t)]	p*
0.0	0.000000	1.00000
1.0	-0.012350	1.00000
2.0	-0.048853	1.00000
3.0	-0.108798	1.00000
4.0	-0.191621	1.00000
5.0	-0.296903	0.99999
6.0	-0.424359	0.99988
7.0	-0.573847	0.99890
8.0	-0.745354	0.99466
9.0	-0.939006	0.98252
10.0	-1.155060	0.95673
11.0	-1.393907	0.91259
12.0	-1.656076	0.84826
13.0	-1.942230	0.76546
14.0	-2.253173	0.66884
15.0	-2.589852	0.56469
16.0	-2.953362	0.45964
17.0	-3.344945	0.35954
18.0	-3.766006	0.26891
19.0	-4.218110	0.19073

p; probability mass* **not** *absorbed ≥ the entry.*

Table VI-2

WEALE (1971) pp.14,15

SQUARE LAW

$a_f = b_f = 0$

$m_A(t) = 9.756,\ m_B(t) = 9.506$

$V[A(t)] = .2440079,\ V[B(t)] = .4942664$

$Cov[A(t),\ B(t)] = -.0123502$

$a_0 = 10, \alpha = 0.05000$

$b_0 = 10, \beta = 0.02500$

$t = 1.0$

$p(a,\ b,\ t) \times 1{,}000{,}000$

b \ a		10	9	8	7	6	5	4	3	2	1	0
		783,725	190,910	23,335	1,908	117	6	0	0	0	0	0
10	512,398	472,367	121,094	15,522	1,326	85	4	0	0	0	0	0
9	301,190	229,160	55,179	6,339	483	27	1	0	0	0	0	0
8	78,439	60,544	12,520	1,284	87	4	0	0	0	0	0	0
7	12,285	10,218	1,385	172	10	0	0	0	0	0	0	0
6	1,523	1,293	212	17	1	0	0	0	0	0	0	0
5	151	131	19	1	0	0	0	0	0	0	0	0
4	13	11	1	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

(a)

WEALE (1971) pp.16,17

SQUARE LAW

$a_f = b_f = 0$

$m_A(t) = 7.945,\ m_B(t) = 5.153$

$V[A(t)] = 2.3096855,\ V[B(t)] = 5.0573893$

$Cov[A(t),\ B(t)] = -1.3482826$

$a_0 = 10,\ \alpha = 0.05000$

$b_0 = 10,\ \beta = 0.02500$

$t = 11.0$

$p(a,\ b,\ t) \times 1{,}000{,}000$

b \ a		10	9	8	7	6	5	4	3	2	1	0
		146,119	261,368	253,736	174,952	95,197	43,233	16,965	5,889	1,839	523	180
10	10,215	261	958	1,756	2,146	1,967	1,442	981	462	212	86	45
9	43,528	1,654	5,457	8,964	9,765	7,928	5,111	2,720	1,227	477	162	64
8	96,381	5,235	15,487	22,705	21,359	15,728	8,876	4,097	1,583	520	146	44
7	146,394	11,047	29,168	37,392	32,468	20,414	10,032	3,393	1,315	362	94	19
6	172,545	17,484	40,980	47,164	35,412	19,428	8,261	2,815	783	179	34	6
5	166,016	22,137	45,765	46,232	30,289	14,393	5,252	1,518	353	56	10	1
4	135,909	23,357	42,259	37,173	21,071	8,594	2,664	646	124	19	2	0
3	97,146	21,123	33,126	25,125	12,195	4,221	1,097	220	34	4	0	0
2	61,773	16,715	22,449	14,501	5,949	1,722	369	60	7	1	0	0
1	35,442	11,758	13,314	7,211	2459	585	101	13	1	0	0	0
0	34,152	15,347	12,405	4,312	1,239	218	28	3	0	0	0	0

(b)

Table VI-3

WEALE (1971) p.18,19

SQUARE LAW

$a_f = b_f = 0$

$m_A(t) = 6.578,\ m_B(t) = 0.631$

$V[A(t) = 7.0306303,\ V\ B(t)] = 2.7311309$

$Cov[A(t),\ B(t)] = -3.2008005$

$a_o = 10,\ \alpha = 0.05000$

$b_o = 10,\ \beta = 0.02500$

$t = 41.0$

$p(a,\ b,\ t) \times 1{,}000{,}000$

a / b		10	9	8	7	6	5	4	3	2	1	0
		93,923	170,519	184,007	156,196	116,825	82,161	56,906	40,075	28,981	21,238	49,170
10	898	0	0	0	0	0	0	0	0	2	7	889
9	3,302	0	0	0	0	0	0	2	7	23	71	3,200
8	6,919	0	0	0	0	1	3	13	46	137	341	6,378
7	10,999	0	0	0	1	4	20	72	214	524	1,054	9,109
6	14,992	0	0	1	5	24	94	298	716	1,438	2,315	10,112
5	18,849	0	0	4	23	102	339	884	1,826	2,982	3,783	8,905
4	22,877	0	2	13	93	350	990	2,160	3,665	4,787	4,678	6,133
3	27,380	1	10	69	312	997	2,370	4,278	5,862	5,982	4,347	3,154
2	32,393	3	37	229	884	2,385	4,708	6,906	7,467	5,750	2,931	1,094
1	37,665	12	127	662	2,152	4,823	7,766	9,047	7,459	4,101	1,319	196
0	823,724	93,907	170,342	183,025	152,726	108,139	65,870	33,256	12,814	3,254	393	0

(a)

WEALE (1971) pp.20,21

SQUARE LAW

$a_f = b_f = 0$

$m_A(t) = 6.486,\ m_B(t) = 0.435$

$V[A(t) = 7.9185709,\ V\ B(t)] = 2.3169327$

$Cov[A(t),\ B(t)] = -2.8030826$

$a_o = 10,\ \alpha = 0.05000$

$b_o = 10,\ \beta = 0.02500$

$t = 80.0$

$p(a,\ b,\ t) \times 1{,}000{,}000$

a / b		10	9	8	7	6	5	4	3	2	1	0
		93,921	170,504	183,914	155,817	115,706	79,602	52,189	32,850	19,621	11,240	84,636
10	896	0	0	0	0	0	0	0	0	0	0	896
9	3,275	0	0	0	0	0	0	0	0	0	0	3,275
8	6,759	0	0	0	0	0	0	0	0	0	0	6,759
7	10,367	0	0	0	0	0	0	0	0	0	1	10,366
6	13,114	0	0	0	0	0	0	0	0	1	7	13,106
5	14,376	0	0	0	0	0	0	0	0	5	42	14,330
4	13,978	0	0	0	0	0	0	0	3	26	181	13,767
3	12,166	0	0	0	0	0	0	2	16	120	603	11,424
2	9,601	0	0	0	0	0	1	9	77	432	1,504	7,578
1	7,314	0	0	0	0	0	5	47	297	1,198	2,630	3,136
0	908,155	93,921	170,504	133,914	155,817	115,706	79,596	52,131	32,456	17,338	6,271	0

(b)

Table VI-4.

WEALE (1971) pp.22,23

SQUARE LAW

$a_f = b_f = 0$

$m_A(t) = 6.482,\ m_B(t) = 0.428$

$V[A(t)] = 7.9600220,\ V[B(t)] = 2.3110105$

$Cov[A(t),\ B(t)] = -2.7767140$

$a_0 = 10,\ \alpha = 0.05000$

$b_0 = 10,\ \beta = 0.02500$

$t = 200.0$

$p(a,\ b,\ t) \times 1{,}000{,}000$

b \ a		10	9	8	7	6	5	4	3	2	1	0
		93,921	170,504	183,914	155,817	115,706	79,601	52,181	32,775	19,125	8,898	87,557
10	896	0	0	0	0	0	0	0	0	0	0	896
9	3,275	0	0	0	0	0	0	0	0	0	0	3,275
8	6,759	0	0	0	0	0	0	0	0	0	0	6,759
7	10,366	0	0	0	0	0	0	0	0	0	0	10,366
6	13,112	0	0	0	0	0	0	0	0	0	0	13,112
5	14,363	0	0	0	0	0	0	0	0	0	0	14,363
4	13,907	0	0	0	0	0	0	0	0	0	0	13,907
3	11,864	0	0	0	0	0	0	0	0	0	0	11,864
2	8,566	0	0	0	0	0	0	0	0	0	0	8,566
1	4,450	0	0	0	0	0	0	0	0	0	1	4,449
0	912,442	93,921	170,504	183,914	155,817	115,706	79,601	52,181	32,775	19,125	8,897	0

Table VI-5

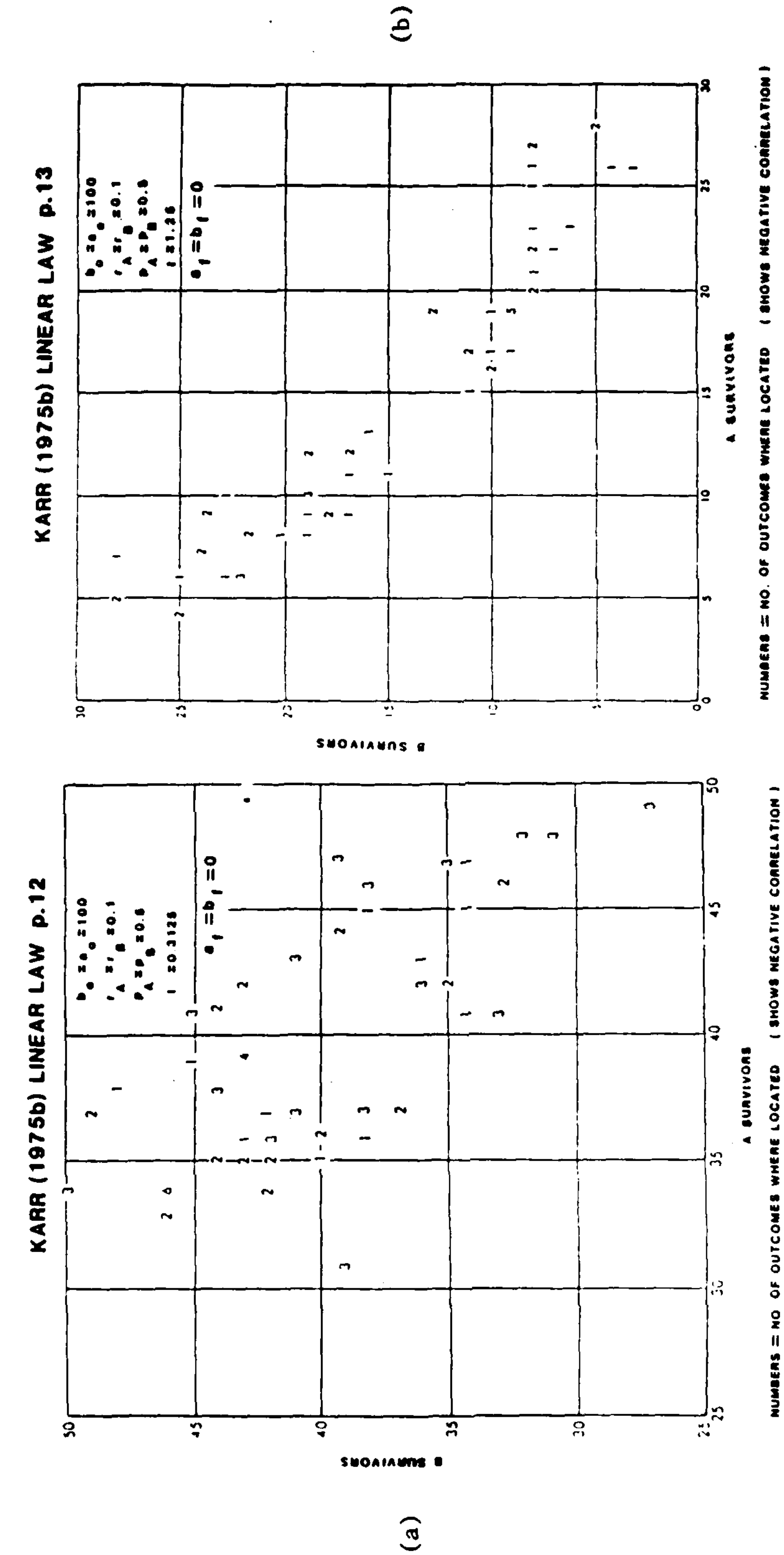
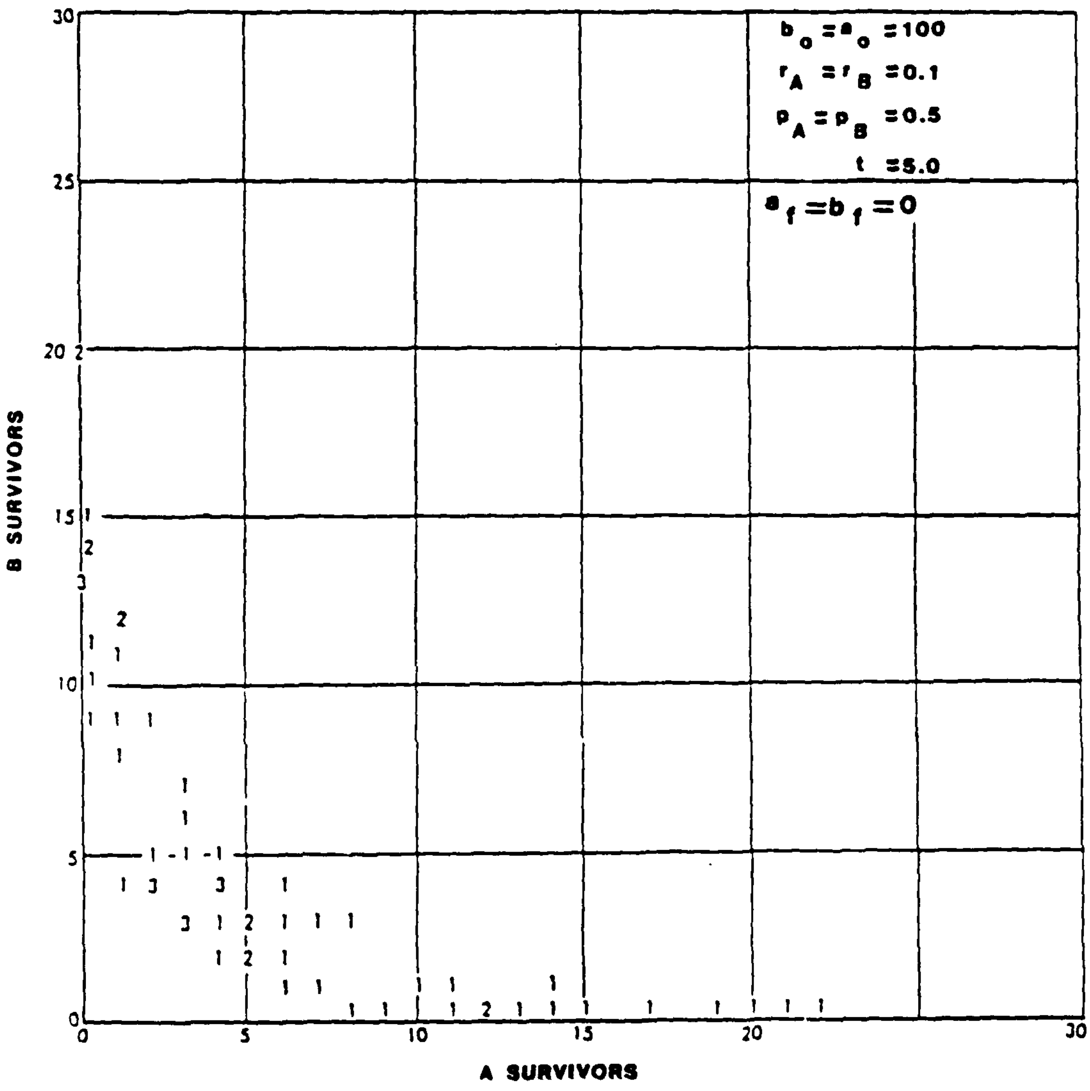

KARR (1975b) LINEAR LAW p.13
B SURVIVORS
A SURVIVORS
NUMBERS = NO. OF OUTCOMES WHERE LOCATED (SHOWS NEGATIVE CORRELATION)
(b)
KARR (1975b) LINEAR LAW p.12
B SURVIVORS
A SURVIVORS
NUMBERS = NO OF OUTCOMES WHERE LOCATED (SHOWS NEGATIVE CORRELATION)
(a)

Figure VI-1

KARR (1975b) LINEAR LAW p.14



NUMBERS = NO. OF OUTCOMES WHERE LOCATED (SHOWS NEGATIVE CORRELATION)

Figure VI-2

WEALE (1975) pp.29,30,31

<u>SPECIAL MODEL</u>

$a_o = b_o = 10$

$a_f = 2,\ b_f = 3$

$E[A(1)] = 9.437,\ E[B(1)] = 5.808$

$V[A(1)] = 0.5544115,\ V[B(1)] = 2.4478500$

$Cov[A(1),\ B(1)] = -0.2098056$

ATTRITION COEFFICIENTS

A Side = a(0.05 + 0.05b)
B Side = b(0.025 + 0.005a)

t = 1.0

p(a, b, 1) x 1,000,000

(a)

b \ a	10	9	8	7	6	5	4	3	2
10	1,930	1,984	951	282	58	9	1	0	0
9	14,350	12,996	5,470	1,418	253	33	3	0	0
8	48,486	38,541	14,181	3,198	493	55	5	0	0
7	98,295	68,242	21,843	4,264	566	54	4	0	0
6	132,849	80,073	22,164	3,724	423	34	2	0	0
5	125,685	65,284	15,513	2,226	214	15	1	0	0
4	84,933	37,670	7,614	925	75	4	0	0	0
3	60,814	18,802	2,734	243	14	1	0	0	0

WEALE (1975) pp.32,33

<u>SPECIAL MODEL</u>

$a_o = b_o = 10$

$a_f = 2,\ b_f = 3$

$E[A(4)] = 9.076,\ E[B(4)] = 3.039$

$V[A(4)] = 1.0768134,\ V[B(4)] = 0.0587164$

$Cov[A(4),\ B(4)] = -0.0711257$

ATTRITION COEFFICIENTS

A Side = a(0.05 + 0.05b)
B Side = b(0.025 + 0.005a)

t = 4.0

p(a, b, 4) x 1,000,000

(b)

b \ a	10	9	8	7	6	5	4	3	2
10	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1
8	0	0	1	3	5	6	5	3	4
7	1	7	20	37	45	40	26	12	9
6	17	86	199	283	273	188	94	34	12
5	186	734	1,339	1,488	1,118	595	229	63	13
4	1,449	4,445	6,298	5,428	3,151	1,289	377	78	10
3	423,822	331,732	149,281	49,466	12,888	2,645	415	47	0

Table VI-6.

(a)

WEALE (1975) pp. 34,35

SPECIAL MODEL

$a_0 = b_0 = 10$

$a_f = 2,\ b_f = 3$

$E[A(8)] = 9.071,\ E[B(8)] = 3.001$

$V[A(8)] = 1.1050641,\ V[B(8)] = 0.0012087$

$Cov[A(8),\ B(8)] = -0.0035915$

ATTRITION COEFFICIENTS

A Side = a(0.05 + 0.05b)
B Side = b(0.025 + 0.005a)

$t = 8.0$

$p(a,\ b,\ 8) \times 1{,}000{,}000$

b \ a	10	9	8	7	6	5	4	3	2
10	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	5
7	0	0	0	0	0	0	0	0	12
6	0	0	0	0	0	0	1	2	27
5	0	0	0	0	2	4	9	12	49
4	0	0	3	10	24	44	58	55	73
3	425,276	336,501	156,653	56,603	17,740	5,111	1,383	341	0

(b)

WEALE (1975) pp. 36,37

SPECIAL MODEL

$a_0 = b_0 = 10$

$a_f = 2,\ b_f = 3$

$E[A(15)] = 9.071,\ E[B(15)] = 3.000$

$V[A(15)] = 1.1055422,\ V[B(15)] = 0.0009021$

$Cov[A(15),\ B(15)] = -0.0024915$

ATTRITION COEFFICIENTS

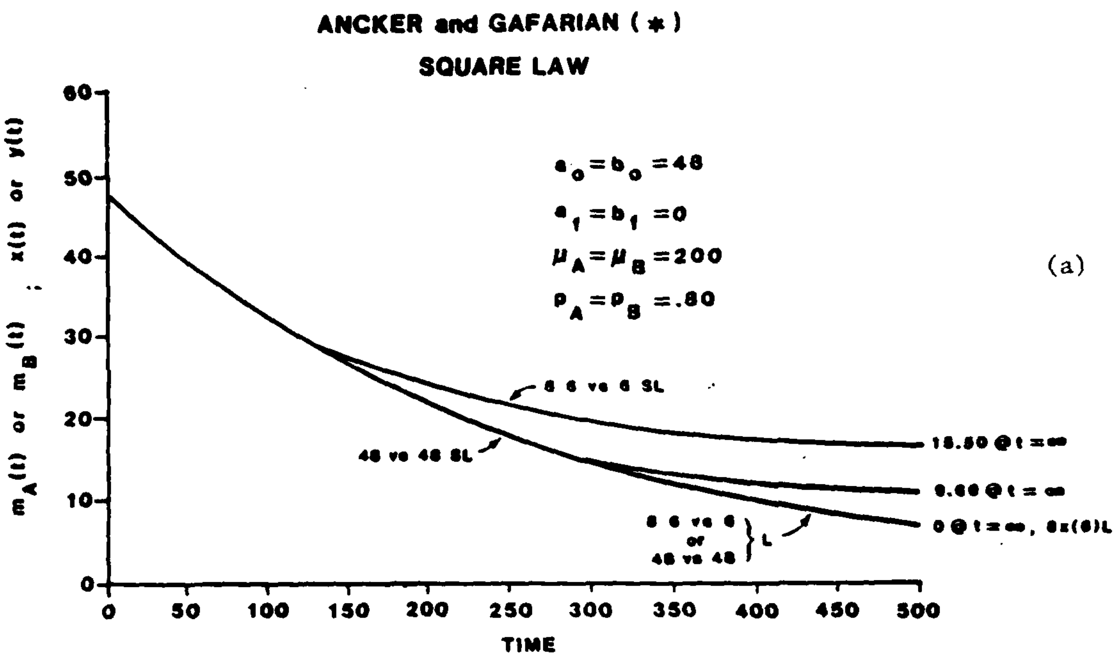
A Side = a(0.05 + 0.05b)
B Side = b(0.025 + 0.005a)

$t = 15.0$

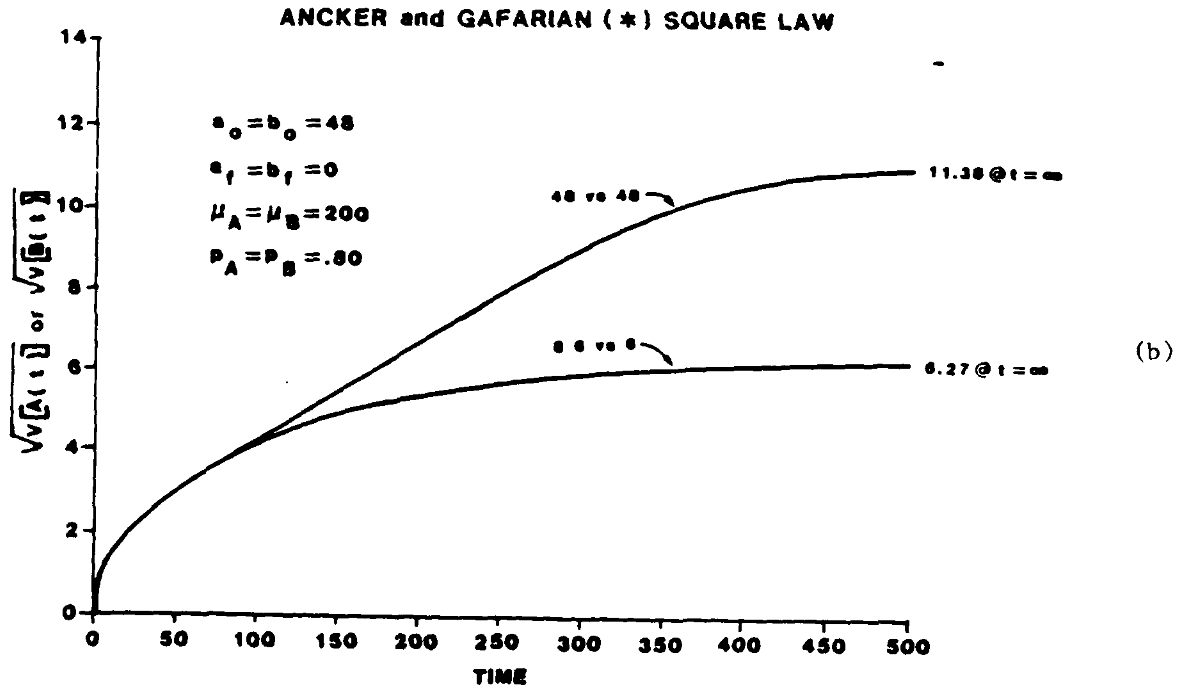
$p(a,\ b,\ 15) \times 1{,}000{,}000$

b \ a	10	9	8	7	6	5	4	3	2
10	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	5
7	0	0	0	0	0	0	0	0	12
6	0	0	0	0	0	0	0	0	27
5	0	0	0	0	0	0	0	0	52
4	0	0	0	0	0	0	0	0	87
3	425,276	336,501	156,656	56,612	17,764	5,156	1,446	406	0

Table VI-7.

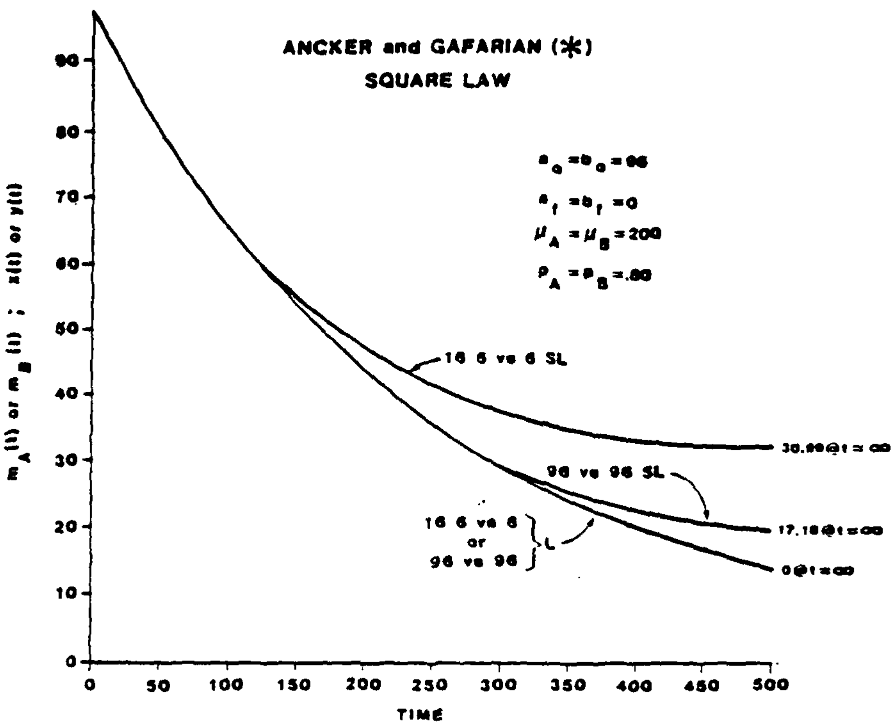


ANCKER and GAFARIAN (*)
SQUARE LAW
$a_o = b_o = 48$
$a_f = b_f = 0$
$\mu_A = \mu_B = 200$
$p_A = p_B = .80$
(a)
TIME

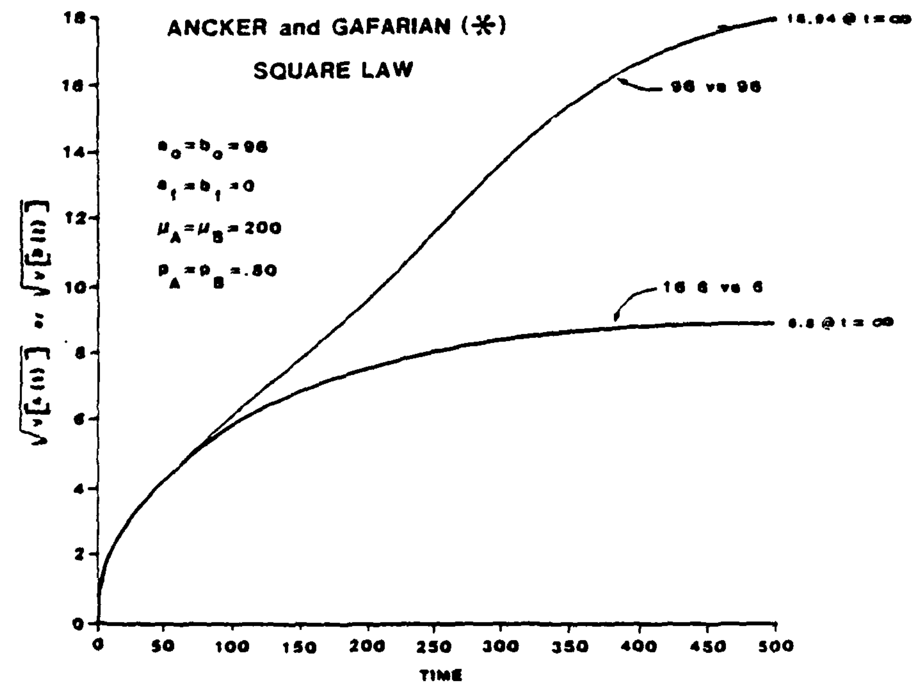


ANCKER and GAFARIAN (*) SQUARE LAW
$a_o = b_o = 48$
$a_f = b_f = 0$
$\mu_A = \mu_B = 200$
$p_A = p_B = .80$
48 vs 48
6.27 @ t = ∞
(b)
TIME

Figure VI-3.

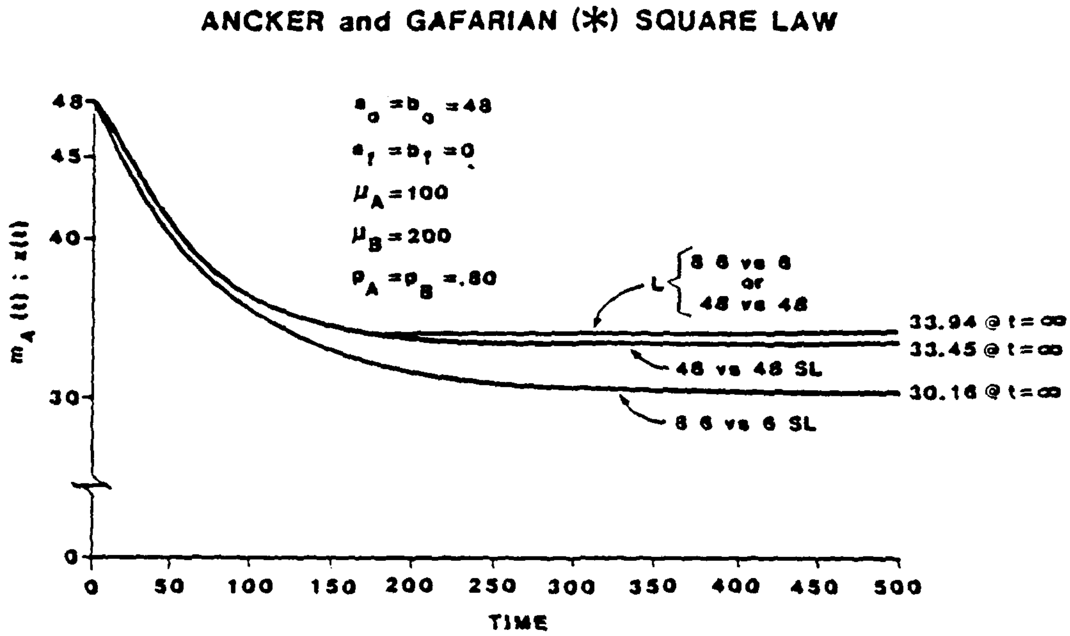


(a)

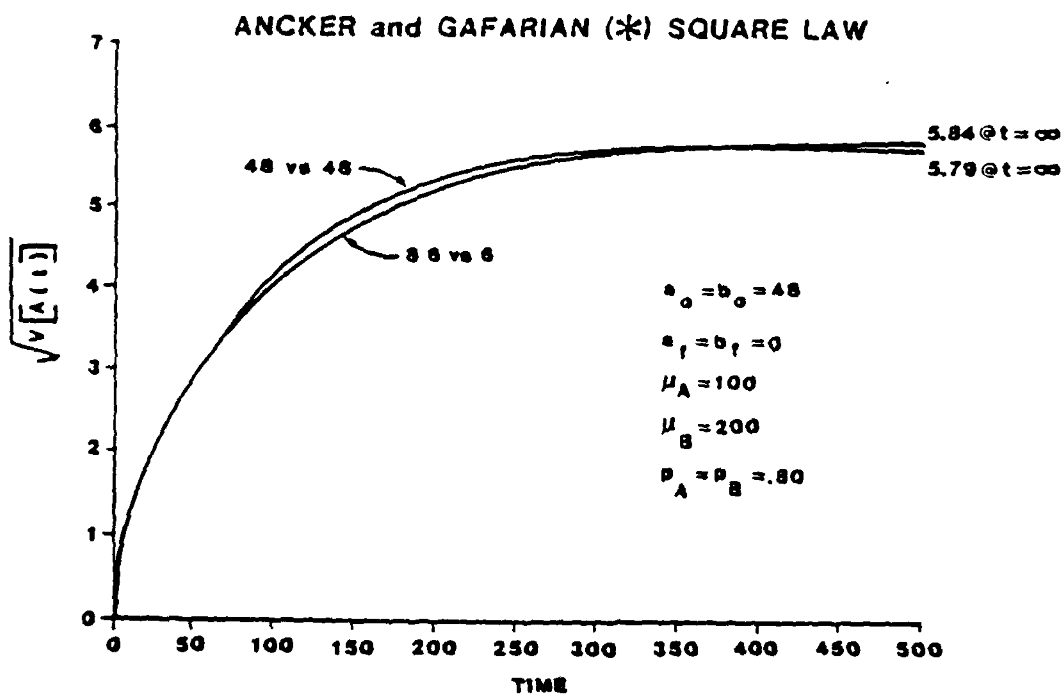


(b)

Figure VI-4

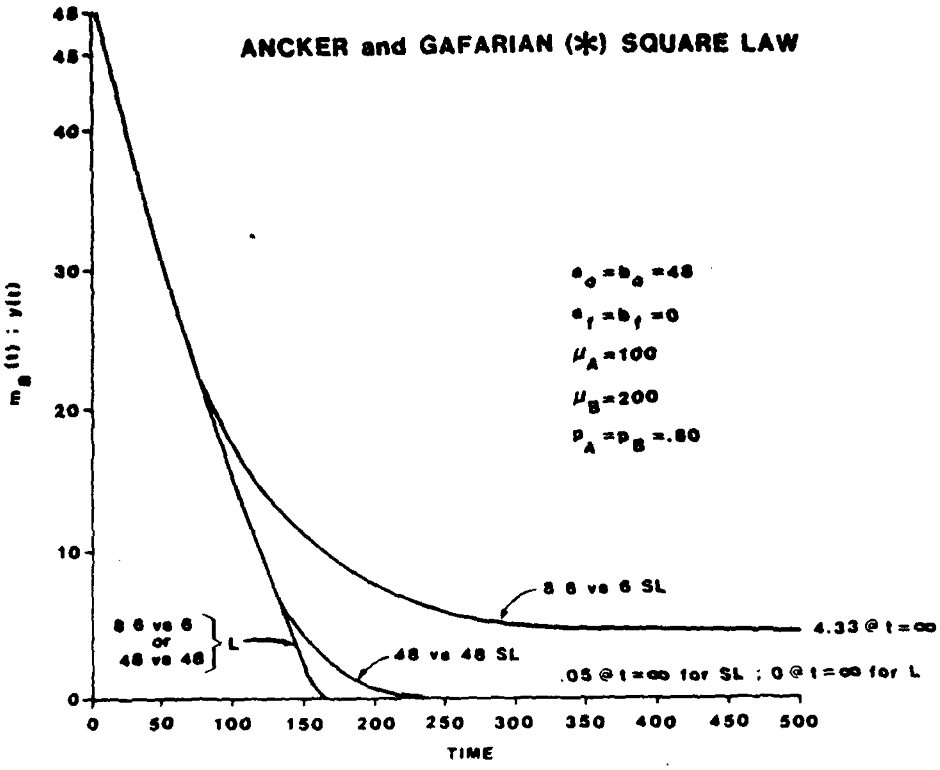


(a)

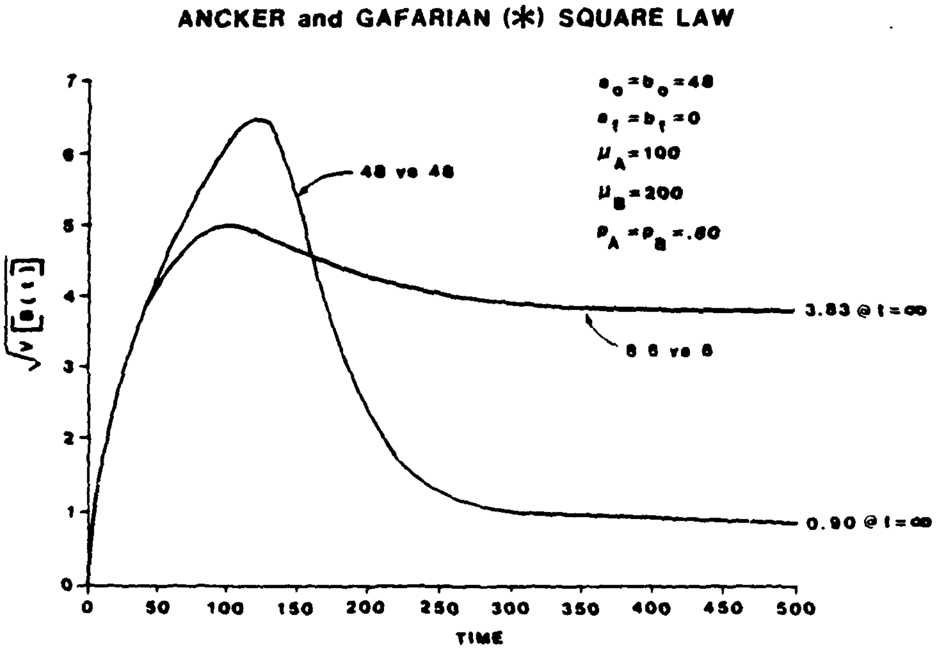


(b)

Figure VI-5

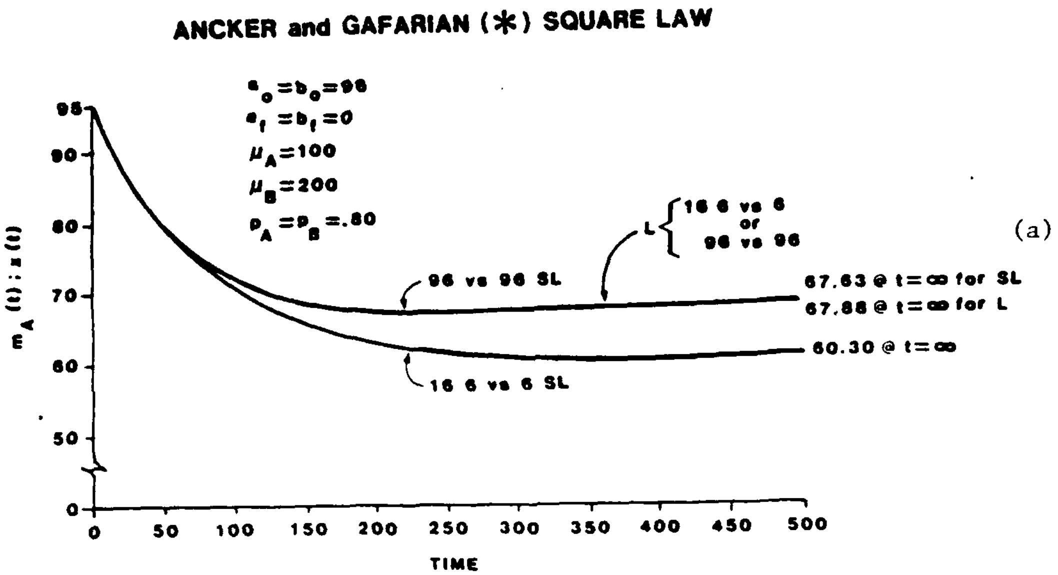


(a)



(b)

Figure VI-6


ANCKER and GAFARIAN (*) SQUARE LAW
$a_o = b_o = 96$
$a_f = b_f = 0$
$\mu_A = 100$
$\mu_B = 200$
$p_A = p_B = .80$
L { 16 6 vs 6 or 96 vs 96
96 vs 96 SL
67.63 @ t=∞ for SL
67.88 @ t=∞ for L
60.30 @ t=∞
16 6 vs 6 SL
$m_A(t) ; x(t)$
TIME
(a)

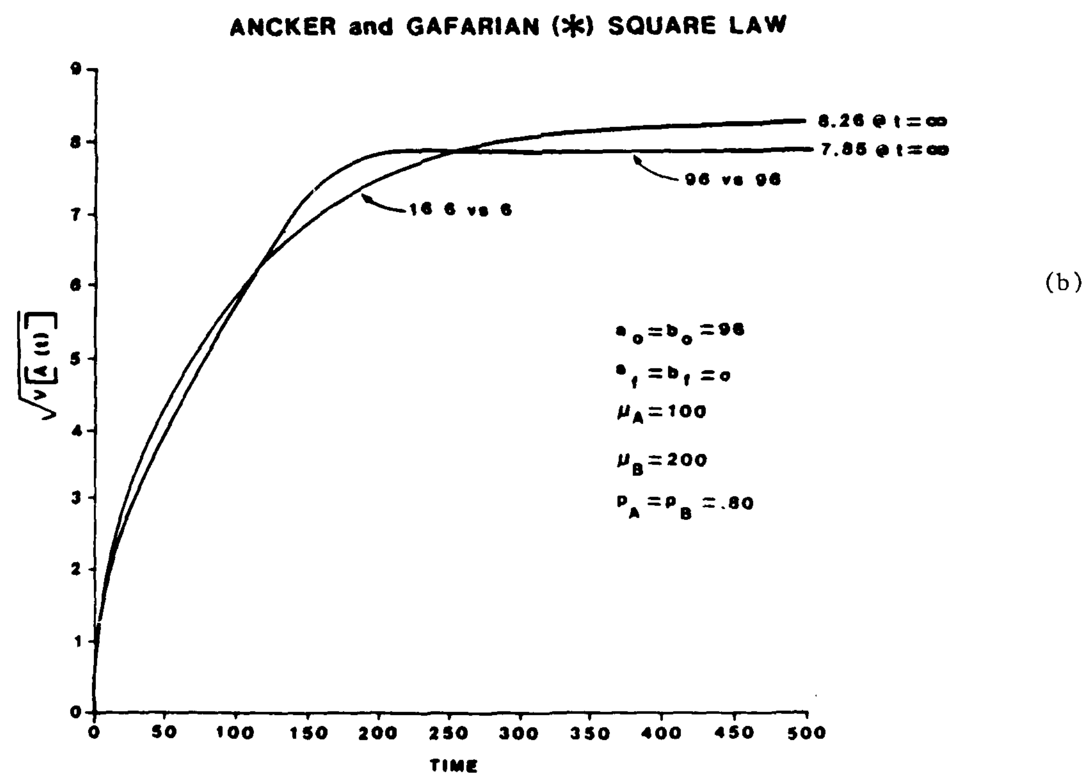

ANCKER and GAFARIAN (*) SQUARE LAW
8.26 @ t=∞
7.85 @ t=∞
96 vs 96
16 6 vs 6
$\sqrt{v[A(t)]}$
$a_o = b_o = 96$
$a_f = b_f = o$
$\mu_A = 100$
$\mu_B = 200$
$p_A = p_B = .80$
TIME
(b)

Figure VI-7.

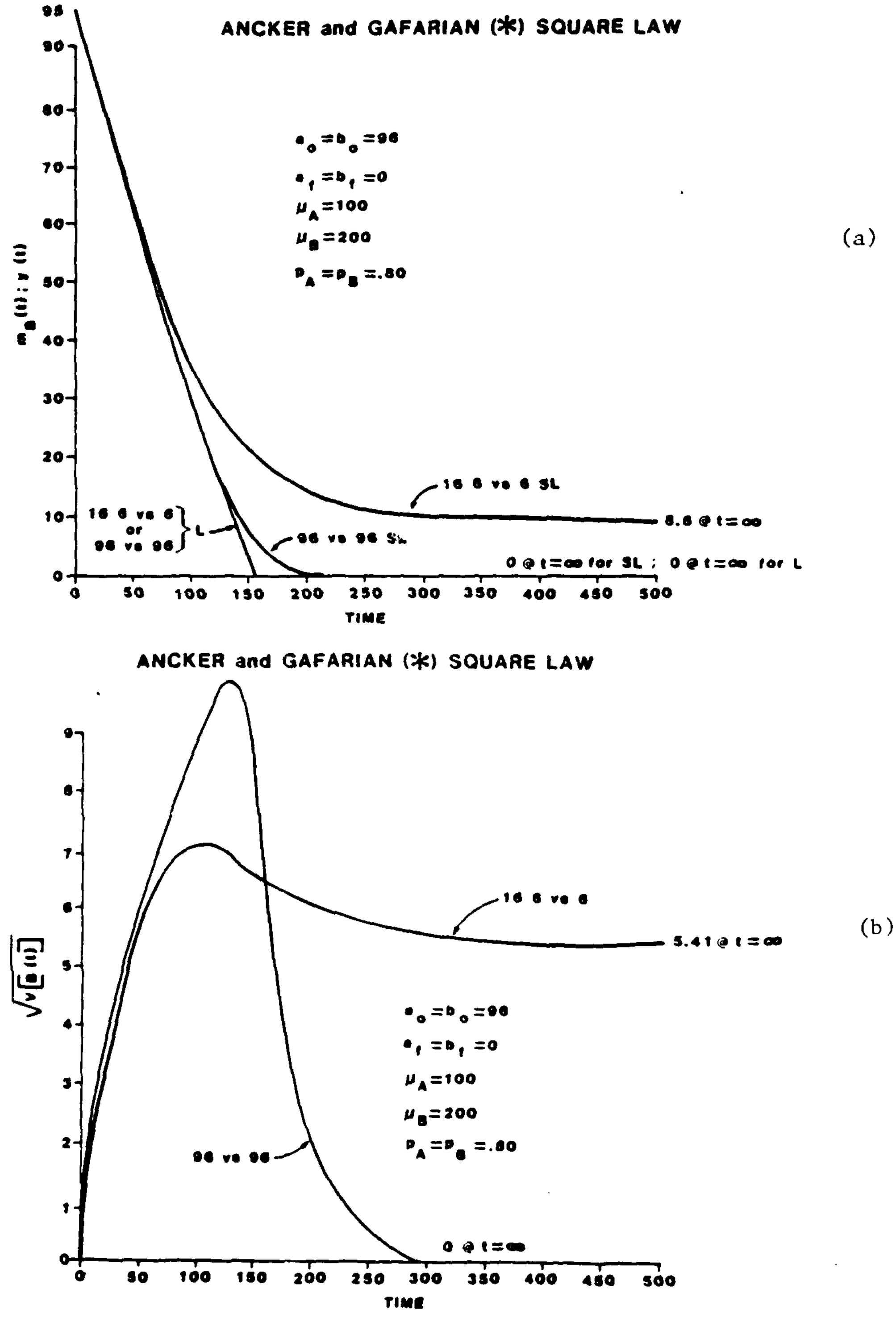

ANCKER and GAFARIAN (*) SQUARE LAW
$a_o = b_o = 96$
$a_f = b_f = 0$
$\mu_A = 100$
$\mu_B = 200$
$p_A = p_B = .80$
(a)
16 6 vs 6 SL
8.6 @ t=∞
16 6 vs 6 or 96 vs 96 } L
96 vs 96 SL
0 @ t=∞ for SL ; 0 @ t=∞ for L
TIME
ANCKER and GAFARIAN (*) SQUARE LAW
16 6 vs 6
5.41 @ t=∞
(b)
$a_o = b_o = 96$
$a_f = b_f = 0$
$\mu_A = 100$
$\mu_B = 200$
$p_A = p_B = .80$
96 vs 96
0 @ t=∞
TIME

Figure VI-8

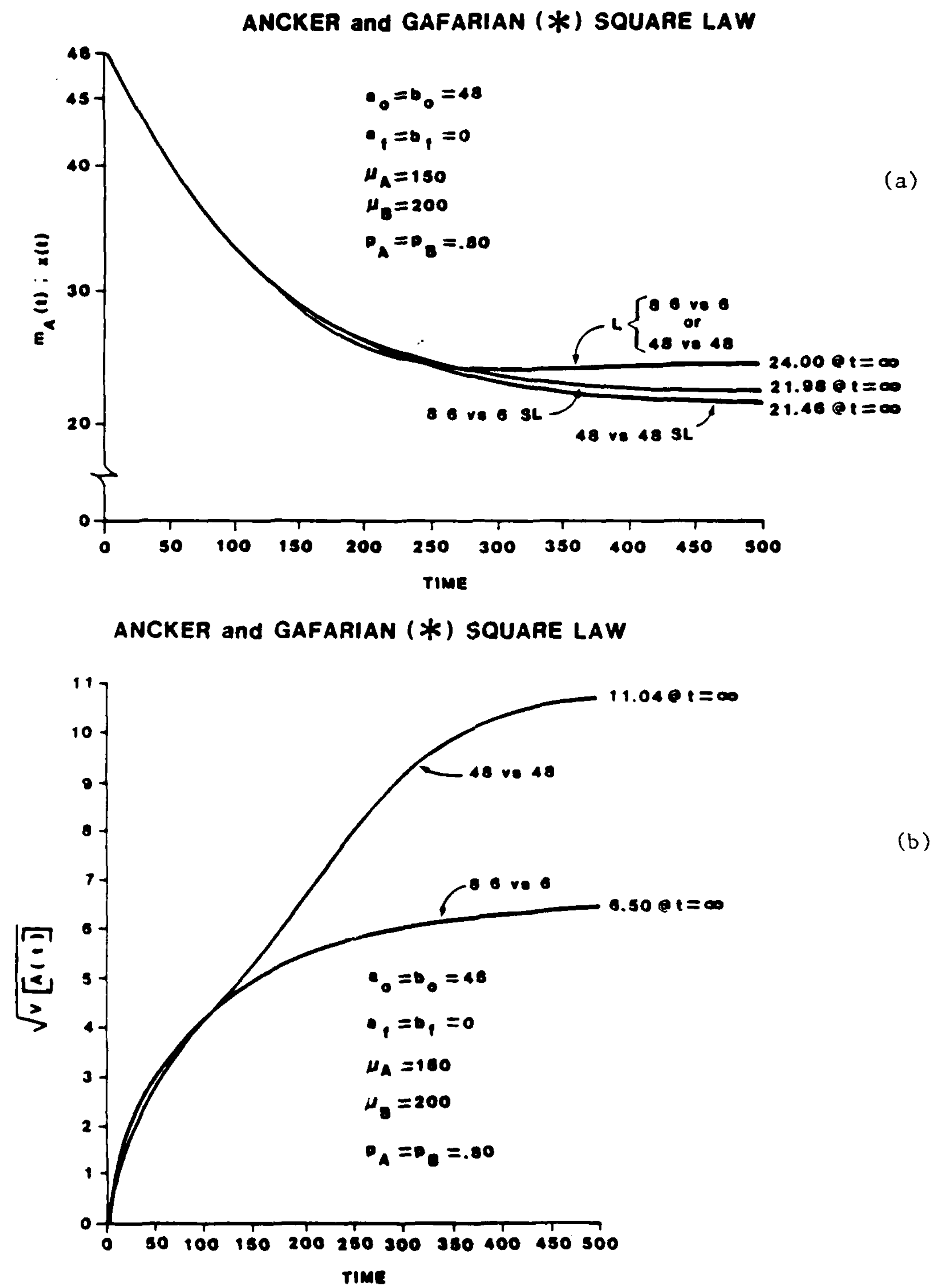
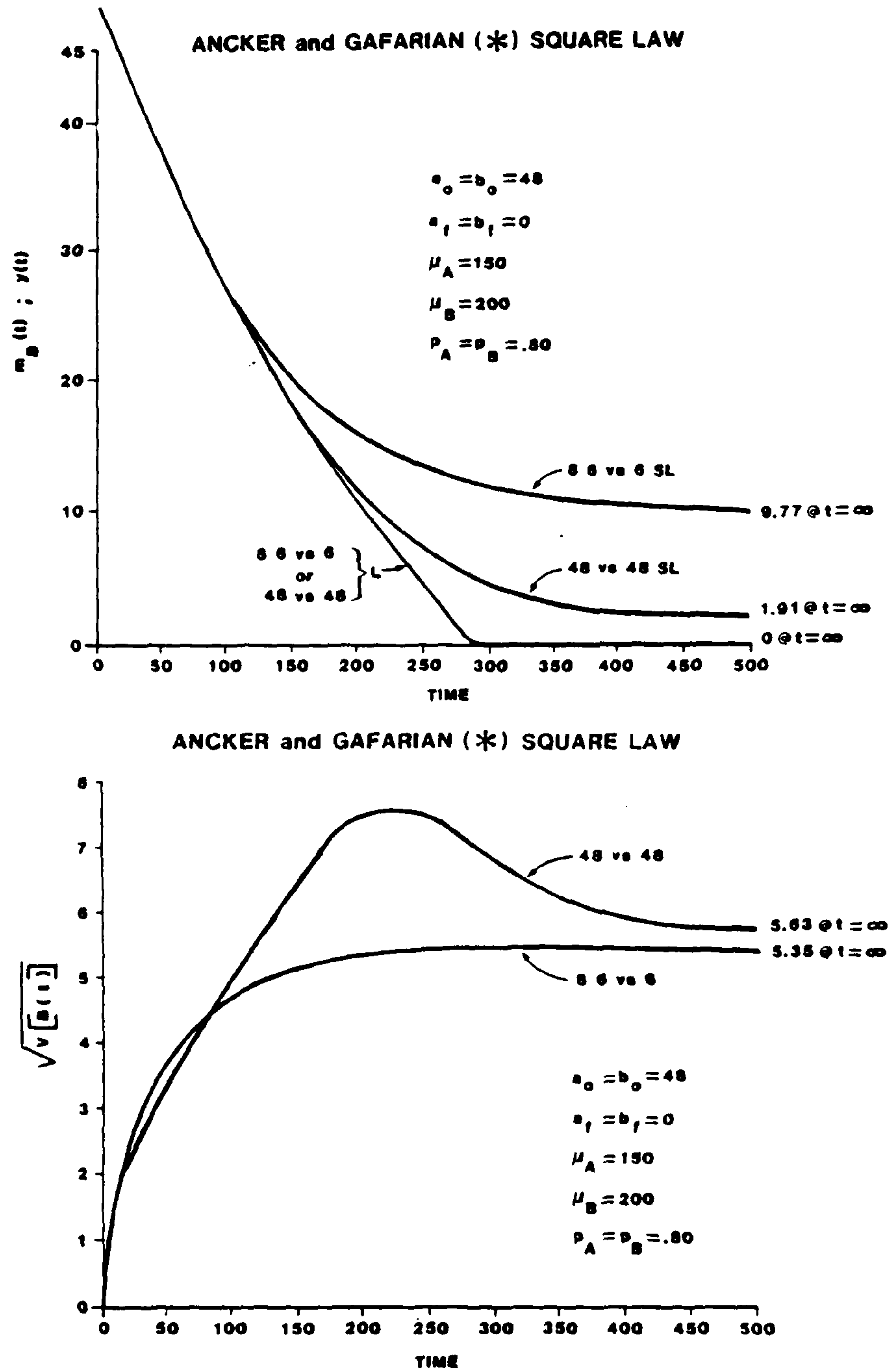

ANCKER and GAFARIAN (*) SQUARE LAW
$a_o = b_o = 48$
$a_f = b_f = 0$
$\mu_A = 150$
$\mu_B = 200$
$p_A = p_B = .80$
(a)
L
8 6 vs 6
or
48 vs 48
24.00 @ t=∞
21.98 @ t=∞
21.46 @ t=∞
8 6 vs 6 SL
48 vs 48 SL
$m_A(t) ; x(t)$
TIME
ANCKER and GAFARIAN (*) SQUARE LAW
11.04 @ t=∞
48 vs 48
(b)
8 6 vs 6
6.50 @ t=∞
$a_o = b_o = 48$
$a_f = b_f = 0$
$\mu_A = 150$
$\mu_B = 200$
$p_A = p_B = .80$
$\sqrt{V[A(t)]}$
TIME

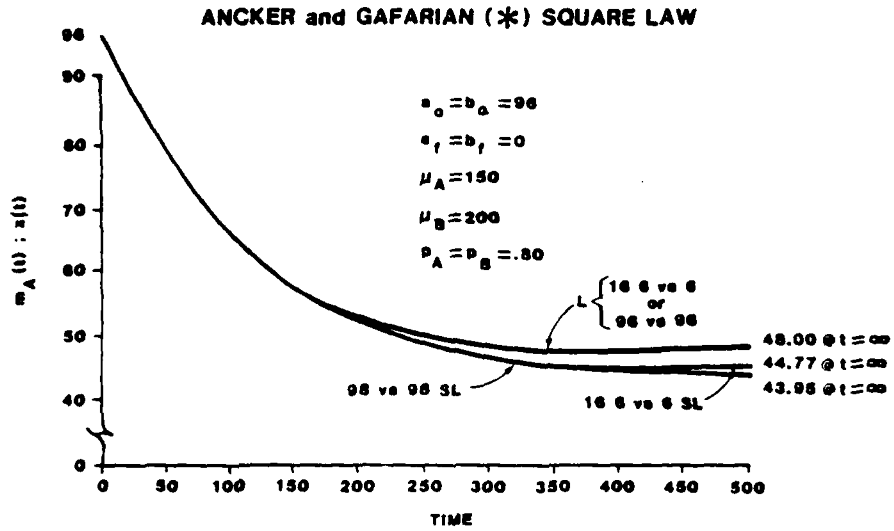
Figure VI-9



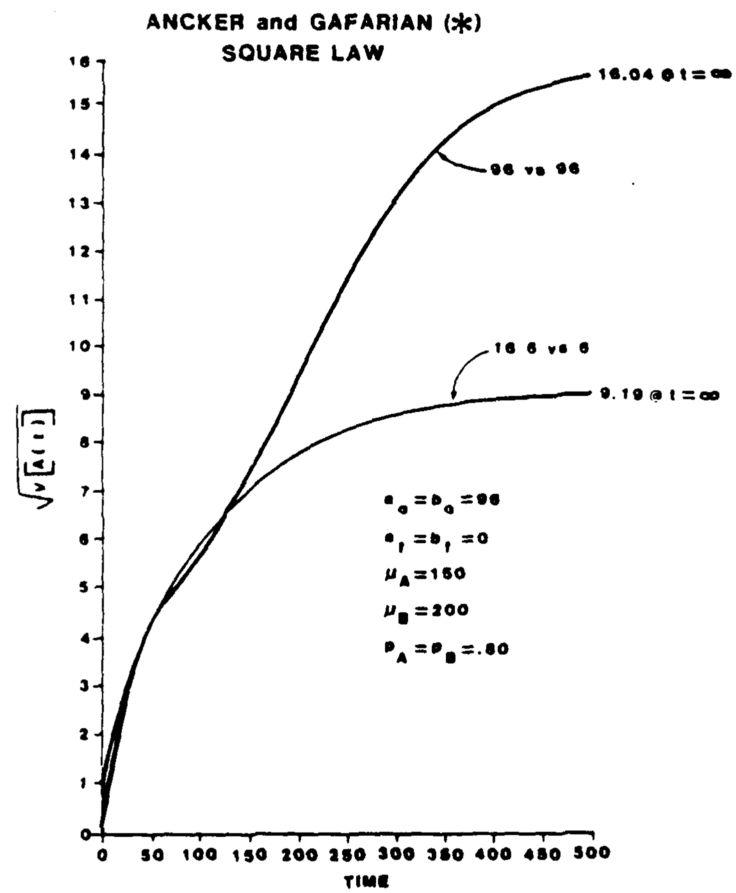
(a)

(b)

Figure VI-10.

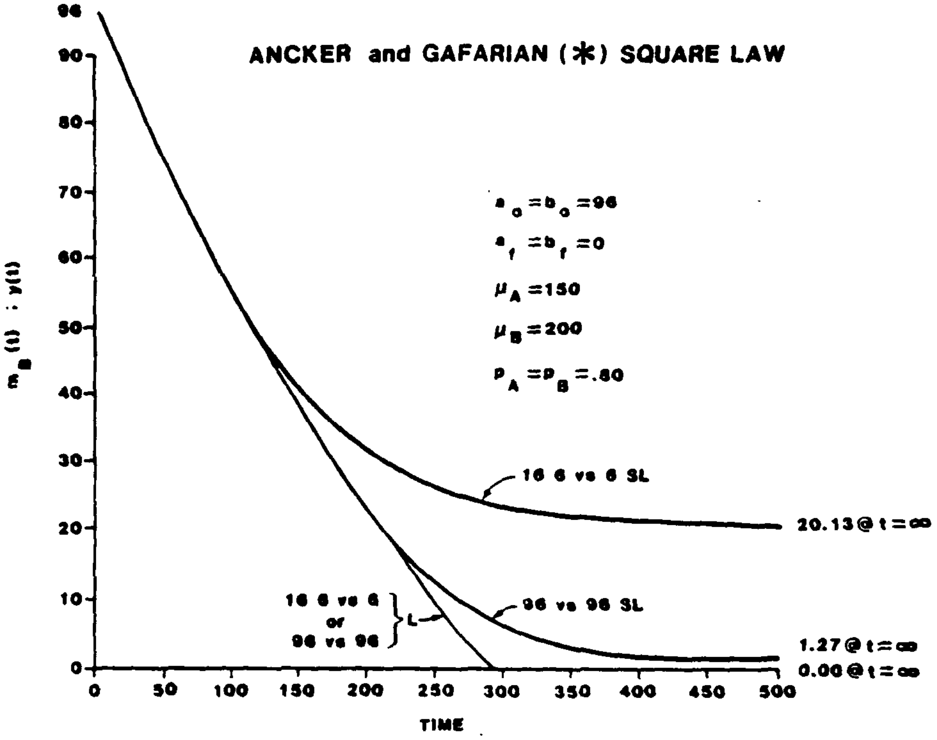


(a)

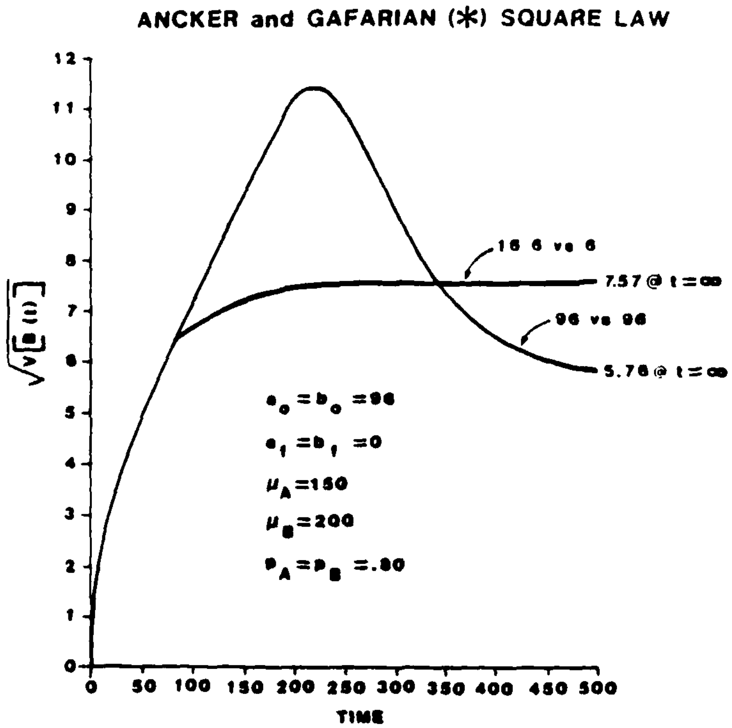


(b)

Figure VI-11



(a)



(b)

Figure VI-12.

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TRAC	TRAC-WSMR-TD-7-88 THE VALIDITY OF ASSUMPTIONS UNDERLYING CURRENT USES OF LANCHESTER ATTRITION RATES	STUDY GIST

THE REASON FOR PERFORMING THE STUDY was to show that many of the prevailing understandings concerning the relationships among the Lanchester, Stochastic Lanchester, and the General Renewal models of combat are erroneous and to collect, organize, and reduce to common notation almost all known tabled results and curves of particular examples.

THE PRINCIPAL RESULTS of this study were:

(1) All Lanchester model and Stochastic Lanchester model mean value equivalent pairs differ for all times except at possible crossing points. These differences may be very large.

(2) At least for the Square Law, the Lanchester model trajectories are neither a universal upper or lower bound of the Stochastic Lanchester model mean value trajectories.

(3) Even near time zero, the Lanchester and Stochastic Lanchester model mean value trajectories may differ considerably (they do not differ materially for the Square Law and the sequence of one-on-one duels version of the Linear Law).

(4) For the Linear Law, the Square Law, the Mixed Law and the Square Law with continuous reinforcements there is a Law of Large Numbers on suitably transformed spaces. However, for untransformed spaces this does not apply, for it can be shown that as the initial force sizes tend to infinity the differences between Lanchester model and Stochastic Lanchester model mean value trajectories tend to zero, or they may even tend to a constant or infinity.

(5) Blackwell's Theorum does not imply that individual combatants with general interfiring times tend to have negative exponentially distributed interfiring times. This is even more strongly the case for terminating processes.

(6) The Palm-Khintchine Theorem does not imply that superposing a large number of combatants with general interfiring times will yield a process with negative exponentially distributed interfiring times. This can only be approximately correct for large numbers and for very large interfiring time means. Again, the Theorem is only valid for non-terminating processes.

(7) Nonhomogeneous Poisson processes do not, in general, approximate general renewal processes.

(8) The Stochastic Lanchester model process variances are generally quite significant and can be important for large force sizes, even near time zero. In addition, general renewal model process variances are significantly different that Stochastic Lanchester model process variances.

(9) The other Lanchester model measures, (a) expected number of survivors, (b) expected time duration of the battle, and (c) probability of winning are even less reliable predictors than the mean value trace.

(10) The basic assumptions of the Stochastic Lanchester models, as well as the general renewal model, cannot hold for large numbers of combatants.

THE MAIN ASSUMPTIONS were that:

(a) All pre-combat decisions have been made and the battle goes forward until terminated by the action itself or by tactical decisions.

(b) The true model which is to be approximated is termed a General Renewal model.

(c) Each marksman fires until he is killed or makes a kill.

(d) The ammunition supply is unlimited.

(e) All fire independently.

THE MAJOR RESTRICTION is that the general renewal model is a superposition of many terminating renewal processes.

THE SCOPE OF THE STUDY is to show that the basic assumptions of current combat models do not hold for large numbers of combatants.

THE STUDY OBJECTIVES are to show that many factors cause large scale battles to be a number of simultaneous and/or sequential smaller scale engagements and that use of current deterministic Lanchester models is incorrect.

THE BASIC APPROACH is two fold. First the Lanchester model applications are examined to show fallacies. Second all known tabled results and curves of particular examples are provided to support the theoretical discussion.

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